

Dynamic programming — recurrences become algorithms

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Practice problems

1. Find the minimum number of coins to make 37 cents using denominations $\{1, 5, 10, 25\}$.
2. A person can climb a staircase of n steps by taking either 1 or 2 steps at a time. Let $dp[i]$ be the number of ways to reach step i . Define the recurrence and find the number of ways to climb 6 steps.
3. Using the coin change logic, find the minimum number of coins to make 13 cents using denominations $\{1, 3, 4\}$.

Solutions

1. Find the minimum number of coins to make 37 cents using denominations $\{1, 5, 10, 25\}$.

- Initialize $dp[0] = 0$ and all other $dp[i] = \infty$.
- Fill the table: $dp[1] = 1, dp[5] = 1, dp[10] = 1, dp[25] = 1$.
- For $dp[37]$, the options are $1 + dp[36], 1 + dp[32], 1 + dp[27], 1 + dp[12]$.
- The best is $1 + dp[12]$ (using a 25c coin).
- Since $dp[12] = 1 + dp[2] = 1 + 2 = 3$ (using a 10c coin and two 1c coins), the total is $1 + 3 = 4$.

Answer: 4

2. A person can climb a staircase of n steps by taking either 1 or 2 steps at a time. Let $dp[i]$ be the number of ways to reach step i . Define the recurrence and find the number of ways to climb 6 steps.

- Base cases: $dp[0] = 1$ (one way to stay at the bottom), $dp[1] = 1$ (one way to take one step).
- Recurrence: To reach step i , you must have come from step $i - 1$ or step $i - 2$. So $dp[i] = dp[i - 1] + dp[i - 2]$.
- Fill the table: $dp[2] = 1 + 1 = 2, dp[3] = 2 + 1 = 3, dp[4] = 3 + 2 = 5, dp[5] = 5 + 3 = 8, dp[6] = 8 + 5 = 13$.
- The number of ways is 13.

Answer: 13

3. Using the coin change logic, find the minimum number of coins to make 13 cents using denominations $\{1, 3, 4\}$.

- Initialize $dp[0] = 0$.
- Fill table: $dp[1] = 1, dp[2] = 2, dp[3] = 1, dp[4] = 1$.
- Continue: $dp[5] = \min(1 + dp[4], 1 + dp[2], 1 + dp[1]) = \min(2, 3, 2) = 2$.
- $dp[6] = \min(1 + dp[5], 1 + dp[3], 1 + dp[2]) = \min(3, 2, 3) = 2$ (two 3c coins).
- $dp[7] = \min(1 + dp[6], 1 + dp[4], 1 + dp[3]) = \min(3, 2, 2) = 2$ (one 3c, one 4c).
- $dp[8] = \min(1 + dp[7], 1 + dp[5], 1 + dp[4]) = \min(3, 3, 2) = 2$ (two 4c coins).
- $dp[9] = \min(1 + dp[8], 1 + dp[6], 1 + dp[5]) = \min(3, 3, 3) = 3$.
- $dp[10] = \min(1 + dp[9], 1 + dp[7], 1 + dp[6]) = \min(4, 3, 3) = 3$.
- $dp[11] = \min(1 + dp[10], 1 + dp[8], 1 + dp[7]) = \min(4, 3, 3) = 3$.
- $dp[12] = \min(1 + dp[11], 1 + dp[9], 1 + dp[8]) = \min(4, 4, 3) = 3$ (three 4c coins).
- $dp[13] = \min(1 + dp[12], 1 + dp[10], 1 + dp[9]) = \min(4, 4, 4) = 4$.

Answer: 4