Stable matching — Gale-Shapley, run live

Lesson 81 · Module 6 — Graph Theory: Structure & Colour · Discrete Mathematics · daily-maths.uk/discrete

Practice problems

1. Given proposers P_1, P_2 and acceptors A, B with preferences $P_1 : [A, B]$, $P_2 : [A, B]$, $A : [P_2, P_1]$, and $B : [P_1, P_2]$, find the stable matching using the proposer-optimal Gale-Shapley algorithm.

2. Suppose we have a stable matching M. If we swap the roles so that the acceptors propose instead of the proposers, will the resulting matching M' always be the same as M? Explain using a small example.

3. Prove that if a proposer is matched with their first choice in some stable matching, they must be matched with their first choice in the proposer-optimal stable matching.

Solutions

1. Given proposers P_1, P_2 and acceptors A, B with preferences $P_1 : [A, B]$, $P_2 : [A, B]$, $A : [P_2, P_1]$, and $B : [P_1, P_2]$, find the stable matching using the proposer-optimal Gale-Shapley algorithm.

   - Step 1: P_1 proposes to A. A is free, so (P_1, A) is a tentative match.
   - Step 2: P_2 proposes to A. A compares P_2 to P_1. Since A prefers P_2, A dumps P_1 and accepts P_2.
   - Step 3: P_1 is now unmatched. P_1 proposes to their next choice, B. B is free, so (P_1, B) is a tentative match.
   - Step 4: All proposers are matched. The algorithm terminates.
   - Step 5: Check for blocking pairs. P_1 is with B (second choice), but A prefers P_2 (current partner) over P_1. P_2 is with A (first choice). No blocking pairs exist.

   Answer: \(P_1, B), (P_2, A)\

2. Suppose we have a stable matching M. If we swap the roles so that the acceptors propose instead of the proposers, will the resulting matching M' always be the same as M? Explain using a small example.

   - Step 1: Consider $P_1 : [A, B], P_2 : [B, A]$ and $A : [P_2, P_1], B : [P_1, P_2]$.
   - Step 2: Proposer-optimal: P_1 proposes to A (accepted), P_2 proposes to B (accepted). Result: $(P_1, A), (P_2, B)$.
   - Step 3: Acceptor-optimal: A proposes to P_2 (accepted), B proposes to P_1 (accepted). Result: $(P_2, A), (P_1, B)$.
   - Step 4: The results are different, showing that the matching depends on who proposes.

   Answer: No, the matching can change depending on who proposes.

3. Prove that if a proposer is matched with their first choice in some stable matching, they must be matched with their first choice in the proposer-optimal stable matching.

   - Step 1: Let M be a stable matching where proposer P is matched with their first choice A.
   - Step 2: Let M_{opt} be the proposer-optimal stable matching produced by Gale-Shapley.
   - Step 3: The proposer-optimal property states that for every proposer P, the partner they are matched with in M_{opt} is at least as preferred as the partner they are matched with in any other stable matching M.
   - Step 4: Since A is P's first choice, no partner can be more preferred than A.
   - Step 5: Therefore, P must be matched with A in M_{opt} to satisfy the proposer-optimal property.

   Answer: By the proposer-optimal property, the partner in the Gale-Shapley matching is the best possible partner across all stable matchings; since the first choice is the absolute best, it must be the partner in the optimal matching.