

Matchings; Hall's marriage theorem — proved

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Practice problems

1. Determine if the following collection of sets has a system of distinct representatives: $S_1 = \{a, b\}$, $S_2 = \{a, b\}$, $S_3 = \{a, c\}$, $S_4 = \{a, b, c\}$.
2. A bipartite graph G has parts X and Y with $|X| = 5$ and $|Y| = 5$. Every vertex in X has degree at least 3, and every vertex in Y has degree at most 3. Does G necessarily have a perfect matching?
3. Suppose we have a bipartite graph G with parts X and Y where $|X| = |Y| = 5$. If the graph is 3-regular, prove that a perfect matching always exists.

Solutions

1. Determine if the following collection of sets has a system of distinct representatives: $S_1 = \{a, b\}$, $S_2 = \{a, b\}$, $S_3 = \{a, c\}$, $S_4 = \{a, b, c\}$.

- Check subsets of size 1: all have size ≥ 1 .
- Check subsets of size 2: any pair has at least 2 elements.
- Check subset $A = \{S_1, S_2, S_3\}$: $N(A) = \{a, b, c\}$, size $3 \geq 3$.
- Check subset $A = \{S_1, S_2, S_3, S_4\}$: $N(A) = \{a, b, c\}$, size $3 \nless 4$.

Answer: No, because the union of all four sets only contains 3 elements, violating Hall's condition for the set of all four sets.

2. A bipartite graph G has parts X and Y with $|X| = 5$ and $|Y| = 5$. Every vertex in X has degree at least 3, and every vertex in Y has degree at most 3. Does G necessarily have a perfect matching?

- The sum of degrees from X is at least $5 \times 3 = 15$.
- The sum of degrees from Y is at most $5 \times 3 = 15$.
- For these to be equal, every vertex in X must have degree exactly 3 and every vertex in Y must have degree exactly 3.
- Thus the graph is 3-regular. In a 3-regular bipartite graph, for any $A \subseteq X$, the number of edges leaving A is $3|A|$.
- These edges must end in $N(A)$. Since each vertex in $N(A)$ has degree 3, the number of edges entering $N(A)$ is at most $3|N(A)|$.
- Thus $3|A| \leq 3|N(A)|$, which implies $|A| \leq |N(A)|$.

Answer: Yes, it must have a perfect matching because it is a 3-regular bipartite graph.

3. Suppose we have a bipartite graph G with parts X and Y where $|X| = |Y| = 5$. If the graph is 3-regular, prove that a perfect matching always exists.

- To prove a perfect matching exists, we check Hall's condition for any $A \subseteq X$.
- The number of edges incident to A is exactly $3|A|$ because the graph is 3-regular.
- All these edges are incident to vertices in $N(A)$.
- Since each vertex in $N(A)$ has degree 3, the total number of edges incident to $N(A)$ is $3|N(A)|$.
- Therefore, $3|A| \leq 3|N(A)|$, which simplifies to $|A| \leq |N(A)|$.
- Since Hall's condition is satisfied for all $A \subseteq X$, a matching saturating X exists. Since $|X| = |Y|$, this is a perfect matching.

Answer: A perfect matching exists because the 3-regularity ensures $|N(A)| \geq |A|$ for all $A \subseteq X$.