The five-colour theorem — proved in full

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Practice problems

1. Suppose a planar graph G has a vertex v of degree 5 whose neighbours v_1, v_2, v_3, v_4, v_5 are coloured Red, Blue, Green, Yellow, and Purple respectively. If the Kempe chain $H(\text{Red}, \text{Green})$ starting at v_1 contains v_3, prove that the Kempe chain $H(\text{Blue}, \text{Yellow})$ starting at v_2 cannot contain v_4.

2. In the proof of the Five Colour Theorem, why is the base case $n \leq 5$ sufficient for the induction to begin?

3. Consider a non-planar graph where a vertex v has degree 5 and its neighbours are coloured with 5 distinct colours. Explain why the Kempe chain argument used in the Five Colour Theorem might fail here.

Solutions

1. Suppose a planar graph G has a vertex v of degree 5 whose neighbours v_1, v_2, v_3, v_4, v_5 are coloured Red, Blue, Green, Yellow, and Purple respectively. If the Kempe chain $H(\text{Red}, \text{Green})$ starting at v_1 contains v_3, prove that the Kempe chain $H(\text{Blue}, \text{Yellow})$ starting at v_2 cannot contain v_4.

   - The vertices v, v_1, v_3 and the path in $H(\text{Red}, \text{Green})$ between v_1 and v_3 form a cycle C.
   - Because the graph is planar, this cycle C divides the plane into an interior and an exterior.
   - Vertex v_2 is on one side of the cycle C, and vertex v_4 is on the other side.
   - Any path from v_2 to v_4 must cross the cycle C.
   - The cycle C consists only of vertices coloured Red or Green.
   - A path in $H(\text{Blue}, \text{Yellow})$ can only contain vertices coloured Blue or Yellow.
   - Therefore, no such path can cross the cycle C, meaning v_2 and v_4 are not connected in $H(\text{Blue}, \text{Yellow})$.

   Answer: The red-green path forms a topological barrier that separates the blue and yellow neighbours.

2. In the proof of the Five Colour Theorem, why is the base case $n \leq 5$ sufficient for the induction to begin?

   - A graph with n vertices can be coloured with n colours by giving each vertex a unique colour.
   - If $n \leq 5$, then n colours are at most 5 colours.
   - Since we are proving that 5 colours are always sufficient, any graph with 5 or fewer vertices is trivially 5-colourable.

   Answer: Because any graph with n vertices is n-colourable, and for $n \leq 5$, $n \leq 5$.

3. Consider a non-planar graph where a vertex v has degree 5 and its neighbours are coloured with 5 distinct colours. Explain why the Kempe chain argument used in the Five Colour Theorem might fail here.

   - The Kempe chain argument relies on the fact that if v_1 and v_3 are connected by a red-green path, then v_2 and v_4 cannot be connected by a blue-yellow path.
   - This relies on the Jordan Curve Theorem, which states that a simple closed curve divides the plane into two regions.
   - In a non-planar graph, edges can 'jump' over each other without intersecting.
   - Therefore, a red-green path between v_1 and v_3 does not necessarily block a blue-yellow path between v_2 and v_4.
   - If both pairs are connected, no single swap of two colours can free up a colour for v.

   Answer: Without planarity, the red-green chain does not act as a barrier to the blue-yellow chain.