

The Four Colour Theorem — the story

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Practice problems

1. Consider a planar graph G with 6 vertices and 9 edges. If G contains a K_4 as a subgraph, what is the minimum possible value for $\chi(G)$ and what is the maximum possible value according to the Four Colour Theorem?
2. Why does the Four Colour Theorem not apply to a graph where vertices represent countries and edges represent any contact, including single points?
3. A map consists of 5 regions where every region shares a border with every other region. Is this map possible to draw on a flat plane? Explain using the Four Colour Theorem and graph theory.

Solutions

1. Consider a planar graph G with 6 vertices and 9 edges. If G contains a K_4 as a subgraph, what is the minimum possible value for $\chi(G)$ and what is the maximum possible value according to the Four Colour Theorem?

- Since G contains K_4 , we know $\chi(G) \geq \chi(K_4) = 4$.
- Since G is planar, the Four Colour Theorem states $\chi(G) \leq 4$.
- Therefore, $\chi(G)$ must be exactly 4.

Answer: Minimum 4, Maximum 4

2. Why does the Four Colour Theorem not apply to a graph where vertices represent countries and edges represent any contact, including single points?

- The theorem specifically applies to planar graphs.
- If point-contacts are edges, the resulting graph may no longer be planar (edges may cross).
- For non-planar graphs, the chromatic number can be arbitrarily large (e.g., K_n requires n colours).

Answer: The resulting graph may be non-planar.

3. A map consists of 5 regions where every region shares a border with every other region. Is this map possible to draw on a flat plane? Explain using the Four Colour Theorem and graph theory.

- If every region shares a border with every other, the corresponding graph is K_5 .
- We know from a previous lesson that K_5 is non-planar.
- Since the graph is non-planar, the map cannot be drawn on a flat plane without borders crossing or regions being disconnected.

Answer: No, because it would require a K_5 graph, which is non-planar.