

Graph colouring II — greedy colouring; Brooks' theorem

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Practice problems

1. Consider a graph with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$. Apply the greedy colouring algorithm using the vertex order 1, 2, 3, 4, 5. How many colours are used?
2. A connected graph G has a maximum degree $\Delta(G) = 4$. If G is not a complete graph K_5 and not an odd cycle, what is the maximum possible value for the chromatic number $\chi(G)$ according to Brooks' theorem?
3. Consider a bipartite graph G with maximum degree $\Delta(G) = 10$. What is the difference between the bound provided by the general greedy rule $\chi(G) \leq \Delta(G) + 1$ and the actual chromatic number $\chi(G)$?

Solutions

1. Consider a graph with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$. Apply the greedy colouring algorithm using the vertex order 1, 2, 3, 4, 5. How many colours are used?
 - Vertex 1 is coloured 1.
 - Vertex 2 is adjacent to 1, so it is coloured 2.
 - Vertex 3 is adjacent to 2, so it is coloured 1.
 - Vertex 4 is adjacent to 3, so it is coloured 2.
 - Vertex 5 is adjacent to 4 (colour 2) and 1 (colour 1), so it must be coloured 3.

Answer: 3

2. A connected graph G has a maximum degree $\Delta(G) = 4$. If G is not a complete graph K_5 and not an odd cycle, what is the maximum possible value for the chromatic number $\chi(G)$ according to Brooks' theorem?
 - Brooks' theorem states that $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle.
 - We are given $\Delta(G) = 4$.
 - Since G is neither K_5 nor an odd cycle, the bound $\chi(G) \leq 4$ applies.

Answer: 4

3. Consider a bipartite graph G with maximum degree $\Delta(G) = 10$. What is the difference between the bound provided by the general greedy rule $\chi(G) \leq \Delta(G) + 1$ and the actual chromatic number $\chi(G)$?
 - The general greedy bound is $\Delta(G) + 1 = 10 + 1 = 11$.
 - By definition, a bipartite graph can be coloured with 2 colours, so $\chi(G) = 2$.
 - The difference is $11 - 2 = 9$.

Answer: 9