

Graph colouring I — chromatic numbers and scheduling

Lesson 76 · Module 6 — Graph Theory: Structure & Colour · Discrete Mathematics · daily-maths.uk/discrete

Practice problems

1. Consider a graph G with 5 vertices forming a cycle C_5 , but with one additional edge connecting two non-adjacent vertices (creating a triangle). What is $\chi(G)$?
2. A company has 6 projects. Some projects share the same lead engineer and cannot be run simultaneously. The conflicts are: (1,2), (1,3), (2,3), (3,4), (4,5), (4,6), (5,6). What is the minimum number of time slots needed?
3. Prove that for any graph G , if $\chi(G) = 2$, then G cannot contain any odd cycles.

Solutions

1. Consider a graph G with 5 vertices forming a cycle C_5 , but with one additional edge connecting two non-adjacent vertices (creating a triangle). What is $\chi(G)$?

- The graph contains a triangle (K_3), so $\omega(G) = 3$, meaning $\chi(G) \geq 3$.
- We attempt to colour the graph with 3 colours.
- The triangle vertices must be colours 1, 2, and 3.
- The remaining two vertices can be coloured using these 3 colours without conflict.
- Since a 3-colouring exists and $\chi(G) \geq 3$, the answer is 3.

Answer: 3

2. A company has 6 projects. Some projects share the same lead engineer and cannot be run simultaneously. The conflicts are: (1,2), (1,3), (2,3), (3,4), (4,5), (4,6), (5,6). What is the minimum number of time slots needed?

- Represent projects as vertices and conflicts as edges.
- Vertices 1,2,3 form a K_3 , so $\chi(G) \geq 3$.
- Vertices 4,5,6 also form a K_3 , so $\chi(G) \geq 3$.
- Colour 1,2,3 with Red, Blue, Green.
- Vertex 4 is connected to 3 (Green), so 4 can be Red.
- Vertices 5 and 6 are connected to 4 (Red) and each other, so they can be Blue and Green.
- A 3-colouring is possible.

Answer: 3

3. Prove that for any graph G , if $\chi(G) = 2$, then G cannot contain any odd cycles.

- Assume G has a proper 2-colouring using colours 1, 2.
- In any path, the colours must alternate: 1, 2, 1, 2, ...
- In a cycle of length k , the k -th vertex is connected to the 1st vertex.
- If k is odd, the k -th vertex will have the same colour as the 1st vertex (both colour 1).
- This creates a conflict, meaning the colouring is not proper.
- Therefore, a 2-colourable graph can only have even cycles.

Answer: A 2-colourable graph is bipartite, and bipartite graphs contain no odd cycles.