Platonic solids from Euler's formula — only five, proved

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Practice problems

1. A regular polyhedron has triangular faces ($n = 3$) and each vertex is shared by 4 edges ($m = 4$). Calculate the number of edges E, vertices V, and faces F.

2. Prove that no Platonic solid can have faces that are regular pentagons ($n = 5$) and vertices where 4 edges meet ($m = 4$).

3. If a regular polyhedron has $V = 20$ and $F = 12$, determine the number of edges E and the values of n and m.

Solutions

1. A regular polyhedron has triangular faces ($n = 3$) and each vertex is shared by 4 edges ($m = 4$). Calculate the number of edges E, vertices V, and faces F.

   - Use the formula $\frac{1}{m} + \frac{1}{n} - \frac{1}{2} = \frac{1}{E}$.
   - Substitute $n = 3$ and $m = 4$: $\frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{3+4-6}{12} = \frac{1}{12}$.
   - Thus, $E = 12$.
   - Use $mV = 2E$: $4V = 24 \implies V = 6$.
   - Use $nF = 2E$: $3F = 24 \implies F = 8$.

   Answer: V=6, E=12, F=8

2. Prove that no Platonic solid can have faces that are regular pentagons ($n = 5$) and vertices where 4 edges meet ($m = 4$).

   - Check the condition $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$.
   - Substitute $n = 5$ and $m = 4$: $\frac{1}{4} + \frac{1}{5} = \frac{5+4}{20} = \frac{9}{20}$.
   - Compare $\frac{9}{20}$ to $\frac{1}{2}$. Since $\frac{1}{2} = \frac{10}{20}$, we see that $\frac{9}{20} < \frac{10}{20}$.
   - The inequality is not satisfied, so such a solid cannot exist.

   Answer: The sum of reciprocals is 9/20, which is less than 1/2.

3. If a regular polyhedron has $V = 20$ and $F = 12$, determine the number of edges E and the values of n and m.

   - Use Euler's formula: $20 - E + 12 = 2 \implies 32 - E = 2 \implies E = 30$.
   - Use $nF = 2E$: $n(12) = 60 \implies n = 5$.
   - Use $mV = 2E$: $m(20) = 60 \implies m = 3$.

   Answer: E=30, n=5, m=3