

Shortest paths — Dijkstra's algorithm, watched step by step

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Practice problems

1. Given a graph with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(1, 2, 10), (1, 3, 3), (3, 2, 4), (2, 4, 2), (3, 4, 8), (3, 5, 1), (4, 5, 6)\}$, find the shortest distance from vertex 1 to vertex 4.
2. In the same graph as Problem 1, what is the shortest path from vertex 1 to vertex 4? List the vertices in order.
3. Suppose a graph has a negative edge weight $w(u, v) = -5$. Explain why the greedy choice in Dijkstra's algorithm might fail to find the shortest path to v .

Solutions

- Given a graph with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(1, 2, 10), (1, 3, 3), (3, 2, 4), (2, 4, 2), (3, 4, 8), (3, 5, 3), (5, 4, 5)\}$, find the shortest distance from vertex 1 to vertex 4.
 - Start at 1: $d(1) = 0$, others ∞ . Relax neighbors: $d(2) = 10, d(3) = 3$.
 - Visit 3 (min distance 3): Relax neighbors: $d(2) = \min(10, 3+4) = 7, d(4) = \min(\infty, 3+8) = 11, d(5) = \min(\infty, 3+3) = 6$.
 - Visit 5 (min distance 6): Relax neighbors: $d(4) = \min(11, 6+5) = 11$.
 - Visit 2 (min distance 7): Relax neighbors: $d(4) = \min(11, 7+2) = 9$.
 - Visit 4 (min distance 9): No unvisited neighbors.

Answer: 9

- In the same graph as Problem 1, what is the shortest path from vertex 1 to vertex 4? List the vertices in order.
 - From the distances in Problem 1, $d(4) = 9$.
 - Check predecessors: $d(4)$ was reached via vertex 5 ($5 + 4 = 9$) or vertex 2 ($7 + 2 = 9$).
 - If via 2: $d(2) = 7$ was reached via vertex 3 ($3 + 4 = 7$). $d(3) = 3$ was reached via vertex 1.
 - Path 1: $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$.
 - If via 5: $d(5) = 6$ was reached via vertex 3 ($3 + 3 = 6$). $d(3) = 3$ was reached via vertex 1.
 - Path 2: $1 \rightarrow 3 \rightarrow 5 \rightarrow 4$.

Answer: 1 \to 3 \to 2 \to 4 \text{ or } 1 \to 3 \to 5 \to 4

- Suppose a graph has a negative edge weight $w(u, v) = -5$. Explain why the greedy choice in Dijkstra's algorithm might fail to find the shortest path to v .
 - Dijkstra's algorithm assumes that once a vertex u is visited, its distance $d(u)$ is the absolute minimum.
 - If there is a negative edge later in the graph, a path with more edges could potentially have a lower total weight than the current minimum.
 - Because Dijkstra's never revisits a 'visited' vertex to update its distance, it will miss any improvements offered by negative edges encountered after the vertex was locked in.

Answer: The greedy assumption that visited vertices have finalized distances is violated by negative weights.