

Spanning Trees; Kruskal's and Prim's Algorithms

Lesson 69 · Module 5 — Graph Theory: Foundations · Discrete Mathematics · dailymaths.uk/discrete

Practice problems

1. Given a graph with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(1, 2, 2), (1, 3, 3), (2, 3, 1), (2, 4, 4), (3, 4, 5), (4, 5, 6)\}$ find the total weight of the minimum spanning tree.
2. In a connected graph where all edge weights are equal to k , what is the total weight of any minimum spanning tree with n vertices?
3. Consider a graph where all edge weights are distinct. If we add a new edge with a weight smaller than any existing edge in the MST, will the MST change? Explain.

Solutions

1. Given a graph with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(1, 2, 2), (1, 3, 3), (2, 3, 1), (2, 4, 4), (3, 4, 5), (4, 5, 1)\}$, find the total weight of the minimum spanning tree.
 - Sort edges: $(2, 3, 1), (4, 5, 1), (1, 2, 2), (1, 3, 3), (2, 4, 4), (3, 4, 5)$.
 - Pick $(2, 3, 1)$. Total weight = 1.
 - Pick $(4, 5, 1)$. Total weight = 2.
 - Pick $(1, 2, 2)$. Total weight = 4.
 - Check $(1, 3, 3)$: adding it creates cycle $(1, 2, 3)$. Discard.
 - Pick $(2, 4, 4)$. Total weight = 8. All vertices connected.

Answer: 8

2. In a connected graph where all edge weights are equal to k , what is the total weight of any minimum spanning tree with n vertices?
 - Any spanning tree of a graph with n vertices must have exactly $n - 1$ edges.
 - Since all edges have the same weight k , every spanning tree has the same total weight.
 - The total weight is the number of edges multiplied by the weight per edge.

Answer: $(n-1)k$

3. Consider a graph where all edge weights are distinct. If we add a new edge with a weight smaller than any existing edge in the MST, will the MST change? Explain.
 - Let the new edge be e with weight $w(e) < w(e')$ for all $e' \in MST$.
 - Adding e to the MST creates a cycle C .
 - Since e is the lightest edge in the graph, it must be the lightest edge in C .
 - Removing any other edge f from C results in a new spanning tree with weight $W(MST) - w(f) + w(e)$.
 - Since $w(e) < w(f)$, the total weight decreases, so the MST must change to include e .

Answer: Yes, the MST will change to include the new edge.