

Trees I — five equivalent definitions

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Practice problems

1. A graph G has 6 vertices and 5 edges. If G is connected, is it a tree? Explain using the definitions learned today.
2. Suppose a graph G with n vertices is acyclic and has $n - 1$ edges. Prove that G must be connected and therefore a tree.
3. Show that if a graph G is a tree, then adding any edge between two non-adjacent vertices u and v creates exactly one cycle.

Solutions

1. A graph G has 6 vertices and 5 edges. If G is connected, is it a tree? Explain using the definitions learned today.

- Step 1: Identify the number of vertices $n = 6$.
- Step 2: Identify the number of edges $|E| = 5$.
- Step 3: Check the condition $|E| = n - 1$. Here, $5 = 6 - 1$, which is true.
- Step 4: Since the graph is connected and has $n - 1$ edges, it satisfies one of the equivalent definitions of a tree.

Answer: Yes, it is a tree.

2. Suppose a graph G with n vertices is acyclic and has $n - 1$ edges. Prove that G must be connected and therefore a tree.

- Step 1: A graph with n vertices and k connected components has at least $n - k$ edges.
- Step 2: For an acyclic graph, the number of edges is exactly $n - k$.
- Step 3: We are given $|E| = n - 1$.
- Step 4: Setting $n - k = n - 1$ gives $k = 1$.
- Step 5: Since there is only one connected component, the graph is connected.

Answer: The graph is connected because $k = 1$.

3. Show that if a graph G is a tree, then adding any edge between two non-adjacent vertices u and v creates exactly one cycle.

- Step 1: Since G is a tree, there exists a unique path P between u and v .
- Step 2: Adding the edge (u, v) creates a cycle consisting of the path P and the new edge (u, v) .
- Step 3: If there were a second cycle, there would have to be another path between u and v in the original tree.
- Step 4: But the path in a tree is unique, so only one cycle can be formed.

Answer: The unique path P plus the new edge forms the only possible cycle.