

Hamiltonian paths — the deceptive twin

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Practice problems

1. A graph has $n = 7$ vertices. Every vertex has a degree of at least 4. Does this graph necessarily contain a Hamiltonian cycle? Justify using Dirac's theorem.
2. Consider a cycle graph C_5 (a pentagon). Does it satisfy the conditions of Dirac's theorem? Is it Hamiltonian?
3. A graph has $n = 6$ vertices. One vertex has degree 1, and the other five have degree 5. Can this graph have a Hamiltonian cycle? Why or why not?

Solutions

1. A graph has $n = 7$ vertices. Every vertex has a degree of at least 4. Does this graph necessarily contain a Hamiltonian cycle? Justify using Dirac's theorem.

- Identify $n = 7$.
- Calculate the Dirac threshold: $n/2 = 7/2 = 3.5$.
- Check if the minimum degree $\delta(G) \geq 3.5$.
- Since $4 \geq 3.5$, the condition is satisfied.

Answer: Yes, it must contain a Hamiltonian cycle.

2. Consider a cycle graph C_5 (a pentagon). Does it satisfy the conditions of Dirac's theorem? Is it Hamiltonian?

- For C_5 , $n = 5$.
- The degree of every vertex is 2.
- The Dirac threshold is $5/2 = 2.5$.
- Since $2 < 2.5$, Dirac's theorem does not apply.
- However, by definition, a cycle graph C_n is its own Hamiltonian cycle.

Answer: It does not satisfy Dirac's theorem, but it is still Hamiltonian.

3. A graph has $n = 6$ vertices. One vertex has degree 1, and the other five have degree 5. Can this graph have a Hamiltonian cycle? Why or why not?

- A Hamiltonian cycle must enter and leave every vertex using two distinct edges.
- This means every vertex in a Hamiltonian cycle must have a degree of at least 2.
- Since one vertex has degree 1, it is impossible to enter and leave it without reusing an edge or vertex.
- Therefore, no Hamiltonian cycle can exist.

Answer: No, because a vertex of degree 1 cannot be part of a cycle.