

Connectivity; components via equivalence relations

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Practice problems

1. Given a graph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (4, 5)\}$, how many connected components does it have? List the vertices in each component.
2. A simple graph has 12 vertices and 8 edges. What is the minimum number of connected components this graph must have?
3. Prove that if a graph G is the disjoint union of two connected graphs G_1 and G_2 , then G is disconnected.

Solutions

1. Given a graph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (4, 5)\}$, how many connected components does it have? List the vertices in each component.

- Start at vertex 1. It is connected to 2, and 2 is connected to 3. This forms the first component: $\{1, 2, 3\}$.
- Look for unvisited vertices. Vertex 4 is unvisited. It is connected to 5. This forms the second component: $\{4, 5\}$.
- All vertices are now visited.

Answer: 2 components: $\{1, 2, 3\}$ and $\{4, 5\}$

2. A simple graph has 12 vertices and 8 edges. What is the minimum number of connected components this graph must have?

- Use the formula $|E| \geq n - k$, where n is the number of vertices and k is the number of components.
- Substitute the known values: $8 \geq 12 - k$.
- Rearrange the inequality to solve for k : $k \geq 12 - 8$.
- This gives $k \geq 4$.

Answer: 4

3. Prove that if a graph G is the disjoint union of two connected graphs G_1 and G_2 , then G is disconnected.

- Let u be a vertex in G_1 and v be a vertex in G_2 .
- By the definition of disjoint union, there are no edges in G that connect a vertex in G_1 to a vertex in G_2 .
- Since any path from u to v would require at least one edge crossing from G_1 to G_2 , no such path exists.
- Because there exists at least one pair of vertices (u, v) with no path between them, G is disconnected by definition.

Answer: The absence of edges between G_1 and G_2 ensures no path exists between their vertices.