Adjacency matrices; counting walks with matrix powers

Practice problems

1. Let G be the cycle graph C_4 with vertices $1, 2, 3, 4$ connected in a ring. Construct the adjacency matrix A and compute $(A^3)_{11}$.

2. For a simple graph G, prove that the number of walks of length 2 from vertex v_i to v_j (where $i \neq j$) is equal to the number of common neighbors of v_i and v_j.

3. A graph G has adjacency matrix A. If the trace of A^2 is 21 and the graph is 3-regular, how many vertices does the graph have?

Solutions

1. Let G be the cycle graph C_4 with vertices $1, 2, 3, 4$ connected in a ring. Construct the adjacency matrix A and compute $(A^3)_{11}$.

   - The adjacency matrix A has $A_{12} = A_{21} = A_{23} = A_{32} = A_{34} = A_{43} = A_{41} = A_{14} = 1$, and all other entries 0.
   - Compute A^2: the diagonal entries are the degrees, so $(A^2)_{ii} = 2$. The off-diagonal entries $(A^2)_{13} = A_{12}A_{23} + A_{14}A_{43} = 1 + 1 = 2$.
   - Compute $(A^3)_{11}$ as the dot product of row 1 of A^2 and column 1 of A. Row 1 of A^2 is $[2, 0, 2, 0]$. Column 1 of A is $[0, 1, 0, 1]^T$.
   - The dot product is $2(0) + 0(1) + 2(0) + 0(1) = 0$.

   Answer: 0

2. For a simple graph G, prove that the number of walks of length 2 from vertex v_i to v_j (where $i \neq j$) is equal to the number of common neighbors of v_i and v_j.

   - A walk of length 2 from v_i to v_j is a sequence $v_i \to v_m \to v_j$.
   - This walk exists if and only if v_m is adjacent to both v_i and v_j.
   - Such a vertex v_m is by definition a common neighbor of v_i and v_j.
   - The number of such walks is the number of such vertices v_m.

   Answer: The number of walks is exactly the number of common neighbors.

3. A graph G has adjacency matrix A. If the trace of A^2 is 21 and the graph is 3-regular, how many vertices does the graph have?

   - We know that $(A^2)_{ii} = \deg(v_i)$.
   - The trace of A^2 is the sum of the degrees of all vertices: $\operatorname{tr}(A^2) = \sum \deg(v_i)$.
   - For a 3-regular graph with n vertices, every vertex has degree 3, so $\operatorname{tr}(A^2) = 3n$.
   - Setting $3n = 21$ gives $n = 7$.

   Answer: 7