

# Graph isomorphism — when are two graphs the same?

Lesson 62 · Module 5 — Graph Theory: Foundations · Discrete Mathematics · [dailymaths.uk/discrete](http://dailymaths.uk/discrete)

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## Practice problems

1. Graph  $G_1$  has vertices  $\{1, 2, 3, 4, 5\}$  and edges  $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3)\}$ . Graph  $G_2$  has vertices  $\{A, B, C, D, E\}$  and edges  $\{(A, B), (B, C), (C, D), (D, E), (E, A), (A, C)\}$ . Are they isomorphic?
2. Two graphs  $G_1$  and  $G_2$  both have 6 vertices and a degree sequence of  $(3, 3, 3, 3, 2, 2)$ .  $G_1$  is connected, but  $G_2$  consists of two disjoint components. Are they isomorphic?
3. Graph  $G_1$  is a cycle  $C_6$ . Graph  $G_2$  is two disjoint cycles  $C_3 \cup C_3$ . Prove they are not isomorphic using a structural invariant other than connectivity.

## Solutions

1. Graph  $G_1$  has vertices  $\{1, 2, 3, 4, 5\}$  and edges  $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3)\}$ . Graph  $G_2$  has vertices  $\{A, B, C, D, E\}$  and edges  $\{(A, B), (B, C), (C, D), (D, E), (E, A), (A, C)\}$ . Are they isomorphic?

- Step 1: Count vertices and edges. Both have 5 vertices and 6 edges.
- Step 2: Compute degree sequences. In  $G_1$ , vertex 1 has degree 3, vertex 3 has degree 3, and others have degree 2. Sequence:  $(3, 3, 2, 2, 2)$ . In  $G_2$ , vertex A has degree 3, vertex C has degree 3, and others have degree 2. Sequence:  $(3, 3, 2, 2, 2)$ .
- Step 3: Try a mapping. Map  $1 \rightarrow A, 3 \rightarrow C, 2 \rightarrow B, 4 \rightarrow D, 5 \rightarrow E$ .
- Step 4: Check edges.  $(1, 2) \rightarrow (A, B)$ ,  $(2, 3) \rightarrow (B, C)$ ,  $(3, 4) \rightarrow (C, D)$ ,  $(4, 5) \rightarrow (D, E)$ ,  $(5, 1) \rightarrow (E, A)$ ,  $(1, 3) \rightarrow (A, C)$ . All edges match.

**Answer:** Yes, they are isomorphic.

2. Two graphs  $G_1$  and  $G_2$  both have 6 vertices and a degree sequence of  $(3, 3, 3, 3, 2, 2)$ .  $G_1$  is connected, but  $G_2$  consists of two disjoint components. Are they isomorphic?

- Step 1: Check basic invariants. Both have 6 vertices and the same degree sequence.
- Step 2: Check connectivity. Connectivity is a graph invariant.
- Step 3: Since  $G_1$  is connected and  $G_2$  is disconnected, they cannot be isomorphic.

**Answer:** No, they are not isomorphic.

3. Graph  $G_1$  is a cycle  $C_6$ . Graph  $G_2$  is two disjoint cycles  $C_3 \cup C_3$ . Prove they are not isomorphic using a structural invariant other than connectivity.

- Step 1: Check degree sequences. Both have all vertices of degree 2.
- Step 2: Look for the smallest cycle (the girth).
- Step 3: In  $G_1$ , the smallest cycle is of length 6.
- Step 4: In  $G_2$ , the smallest cycle is of length 3.
- Step 5: Since the length of the shortest cycle is an invariant and they differ, the graphs are not isomorphic.

**Answer:** Not isomorphic because  $G_2$  contains  $C_3$  while  $G_1$  does not.