The handshake lemma — first graph proofs

Practice problems

1. Does a graph exist with six vertices having degrees $4, 4, 3, 3, 2, 2$?
2. A graph has 15 edges. If 5 vertices have degree 4, what is the average degree of the remaining vertices if there are 10 vertices in total?
3. Prove that in any group of people, the number of people who have shaken hands an odd number of times is always even.

Solutions

1. Does a graph exist with six vertices having degrees $4, 4, 3, 3, 2, 2$?

   - First, check the number of odd-degree vertices. The degrees are 3 and 3, so there are 2 odd-degree vertices. Since 2 is even, the parity condition is satisfied.
   - Next, calculate the sum of degrees: $4 + 4 + 3 + 3 + 2 + 2 = 18$.
   - Apply the Handshake Lemma: $2|E| = 18$, which means $|E| = 9$.
   - Since the sum is even and the number of odd vertices is even, such a graph can exist.

   Answer: Yes

2. A graph has 15 edges. If 5 vertices have degree 4, what is the average degree of the remaining vertices if there are 10 vertices in total?

   - Total sum of degrees is $2 \times |E| = 2 \times 15 = 30$.
   - The sum of degrees for the first 5 vertices is $5 \times 4 = 20$.
   - The remaining sum for the other 5 vertices is $30 - 20 = 10$.
   - The average degree of the remaining 5 vertices is $10/5 = 2$.

   Answer: 2

3. Prove that in any group of people, the number of people who have shaken hands an odd number of times is always even.

   - Represent the group as a graph where vertices are people and edges are handshakes.
   - The number of handshakes a person has made is the degree of their vertex.
   - By the Handshake Lemma, $\sum \deg(v) = 2|E|$, so the total sum of degrees is even.
   - The sum of degrees of even-degree vertices is always even.
   - Therefore, the sum of degrees of odd-degree vertices must also be even to keep the total sum even.
   - A sum of odd numbers is even if and only if there is an even number of terms.

   Answer: The number of odd-degree vertices must be even.