# Graphs: the mathematics of connection



## Practice problems

1. Given $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 3\}\}$, is this a simple graph? List the degree of each vertex.

2. Consider a directed graph with $V = \{A, B, C\}$ and $E = \{(A, B), (B, C), (C, A), (A, A)\}$. Find the in-degree and out-degree of vertex $A$.

3. A multigraph has 4 vertices. Vertex 1 has degree 3, Vertex 2 has degree 3, Vertex 3 has degree 2, and Vertex 4 has degree 2. Is it possible for this graph to be a simple graph if Vertex 1 is connected to all other vertices?

## Solutions

1. Given $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 3\}\}$, is this a simple graph? List the degree of each vertex.

   - Check for loops: No edge is of the form $\{u, u\}$.
   - Check for multiple edges: Each pair in $E$ is unique.
   - Since there are no loops and no multiple edges, it is a simple graph.
   - Count edges for each vertex: 1 is in 2 edges, 2 is in 2 edges, 3 is in 4 edges, 4 is in 2 edges, 5 is in 2 edges.

   **Answer:** Yes, it is a simple graph. $\deg(1) = 2, \deg(2) = 2, \deg(3) = 4, \deg(4) = 2, \deg(5) = 2$.

2. Consider a directed graph with $V = \{A, B, C\}$ and $E = \{(A, B), (B, C), (C, A), (A, A)\}$. Find the in-degree and out-degree of vertex $A$.

   - In-degree of $A$: Count edges where $A$ is the second element. $(C, A)$ and $(A, A)$ both end at $A$.
   - Out-degree of $A$: Count edges where $A$ is the first element. $(A, B)$ and $(A, A)$ both start at $A$.
   - The loop $(A, A)$ contributes to both.

   **Answer:** $\text{in-deg}(A) = 2, \text{out-deg}(A) = 2$.

3. A multigraph has 4 vertices. Vertex 1 has degree 3, Vertex 2 has degree 3, Vertex 3 has degree 2, and Vertex 4 has degree 2. Is it possible for this graph to be a simple graph if Vertex 1 is connected to all other vertices?

   - If Vertex 1 is connected to all other vertices (2, 3, and 4) in a simple graph, it must have exactly one edge to each.
   - This would give Vertex 1 a degree of 3, which matches the prompt.
   - Now check Vertex 2. It has degree 3. It is already connected to Vertex 1. It needs 2 more edges.
   - The only remaining vertices are 3 and 4. If it connects to both, it reaches degree 3.
   - Now check Vertex 3. It is connected to 1 and 2. Its degree is now 2. It cannot have any more edges.
   - Now check Vertex 4. It is connected to 1 and 2. Its degree is now 2. It cannot have any more edges.
   - This configuration satisfies all degree requirements without loops or multiple edges.

   **Answer:** Yes, it is possible.

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