

Module 4 Review and Self-Test

Lesson 58 · Module 4 — Recurrences & Generating Functions · Discrete Mathematics · daily-maths.uk/discrete

Practice problems

1. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 4$.
2. Find the growth rate of $T(n) = 8T(n/2) + n^2$ using the Master Theorem.
3. Find the number of strings of length n using letters X, Y where X must appear an even number of times, using exponential generating functions.

Solutions

1. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 4$.

- The characteristic equation is $r^2 - 4r + 4 = 0$.
- This factors as $(r - 2)^2 = 0$, so there is a repeated root $r = 2$.
- The general solution is $a_n = (A + Bn)2^n$.
- Using $a_0 = 1$, we get $A(2^0) = 1$, so $A = 1$.
- Using $a_1 = 4$, we get $(1 + B)2^1 = 4$, which means $1 + B = 2$, so $B = 1$.
- The closed form is $a_n = (1 + n)2^n$.

Answer: $a_n = (n+1)2^n$

2. Find the growth rate of $T(n) = 8T(n/2) + n^2$ using the Master Theorem.

- Identify $a = 8, b = 2, f(n) = n^2$.
- Compute the critical value $n^{\log_b a} = n^{\log_2 8} = n^3$.
- Compare $f(n) = n^2$ with n^3 .
- Since $n^2 = O(n^{3-\epsilon})$ for $\epsilon = 1$, we are in Case 1.
- The growth rate is $\Theta(n^{\log_b a})$.

Answer: $T(n) = \Theta(n^3)$

3. Find the number of strings of length n using letters X, Y where X must appear an even number of times, using exponential generating functions.

- The EGF for X is $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
- The EGF for Y is e^x .
- The total EGF is $E(x) = \frac{e^x + e^{-x}}{2} e^x = \frac{e^{2x} + 1}{2}$.
- The coefficient of $\frac{x^n}{n!}$ is the n -th term a_n .
- For $n > 0$, the constant term 1 does not contribute, so $a_n = \frac{2^n}{2} = 2^{n-1}$.

Answer: $a_n = 2^{n-1}$ for $n \geq 0$