

Recurrence gauntlet — mixed practice

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Practice problems

1. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2} + 3$ with $a_0 = 1$ and $a_1 = 2$.
2. Determine the asymptotic growth of $T(n) = 2T(n/4) + \sqrt{n}$ using the Master Theorem.
3. Solve the labelled recurrence $a_n = na_{n-1} + n!$ with $a_0 = 1$.

Solutions

1. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2} + 3$ with $a_0 = 1$ and $a_1 = 2$.

- Find the homogeneous solution: $r^2 - 4r + 4 = 0$ gives a repeated root $r = 2$, so $a_n^{(h)} = (A + Bn)2^n$.
- Find a particular solution: try $a_n^{(p)} = C$. Substituting into the recurrence: $C = 4C - 4C + 3$, so $C = 3$.
- General solution: $a_n = (A + Bn)2^n + 3$.
- Use initial conditions: $a_0 = A + 3 = 1 \implies A = -2$.
- Use $a_1 = (A + B)2 + 3 = 2 \implies (-2 + B)2 = -1 \implies B = 1.5$.
- Final form: $a_n = (-2 + 1.5n)2^n + 3$.

Answer: $a_n = (1.5n - 2)2^n + 3$

2. Determine the asymptotic growth of $T(n) = 2T(n/4) + \sqrt{n}$ using the Master Theorem.

- Identify parameters: $a = 2, b = 4, f(n) = n^{1/2}$.
- Compute critical value: $n^{\log_4 2} = n^{1/2}$.
- Compare $f(n)$ and critical value: $n^{1/2}$ is exactly the same as $n^{1/2}$.
- This is Case 2 of the Master Theorem.
- The solution is $T(n) = \Theta(n^{1/2} \log n)$.

Answer: $\Theta(\sqrt{n} \log n)$

3. Solve the labelled recurrence $a_n = na_{n-1} + n!$ with $a_0 = 1$.

- Divide by $n!$: $\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + 1$.
- Let $b_n = \frac{a_n}{n!}$. Then $b_n = b_{n-1} + 1$ with $b_0 = 1$.
- This is an arithmetic progression: $b_n = b_0 + n(1) = 1 + n$.
- Substitute back: $\frac{a_n}{n!} = n + 1$.
- Solve for a_n : $a_n = (n + 1)n! = (n + 1)!$.

Answer: $a_n = (n+1)!$