

Exponential Generating Functions

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Practice problems

1. Find the number of strings of length n using the alphabet $\{A, B\}$ such that the letter A appears an odd number of times.
2. Find the E G F for the sequence $a_n = n^2$.
3. How many ways can we form a sequence of length n using letters A, B, C such that A appears an even number of times and B appears an even number of times?

Solutions

1. Find the number of strings of length n using the alphabet $\{A, B\}$ such that the letter A appears an odd number of times.

- The E G F for A (odd counts) is $\sinh(x) = \frac{e^x - e^{-x}}{2}$.
- The E G F for B (any count) is e^x .
- The total E G F is $E(x) = \frac{e^x - e^{-x}}{2} e^x = \frac{e^{2x} - 1}{2}$.
- The coefficient of $x^n/n!$ in e^{2x} is 2^n .
- The coefficient of $x^n/n!$ in 1 is 1 if $n = 0$ and 0 otherwise.
- For $n > 0$, the number of strings is $\frac{2^n}{2} = 2^{n-1}$.

Answer: 2^{n-1} \text{ for } n \geq 0

2. Find the E G F for the sequence $a_n = n^2$.

- We know $e^x = \sum \frac{x^n}{n!}$.
- Differentiating once gives $e^x = \sum \frac{nx^{n-1}}{n!}$.
- Multiplying by x gives $xe^x = \sum \frac{nx^n}{n!}$.
- Differentiating again gives $(1+x)e^x = \sum \frac{n^2x^{n-1}}{n!}$.
- Multiplying by x again gives $(x+x^2)e^x = \sum \frac{n^2x^n}{n!}$.

Answer: $(x+x^2)e^x$

3. How many ways can we form a sequence of length n using letters A, B, C such that A appears an even number of times and B appears an even number of times?

- The E G F for A is $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
- The E G F for B is $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
- The E G F for C is e^x .
- The total E G F is $E(x) = \frac{(e^x + e^{-x})^2}{4} e^x = \frac{e^{2x} + 2 + e^{-2x}}{4} e^x$.
- Simplify: $E(x) = \frac{1}{4}(e^{3x} + 2e^x + e^{-x})$.
- Extract the coefficient of $x^n/n!$: $a_n = \frac{3^n + 2(1)^n + (-1)^n}{4}$.

Answer: $\frac{3^n + 2 + (-1)^n}{4}$