

The Catalan generating function

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Practice problems

1. Use the binomial expansion of $(1 - 4x)^{1/2}$ to find the coefficient of x^3 in $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.
2. Show that the function $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ satisfies the equation $C(x) = 1 + xC(x)^2$.
3. Determine the radius of convergence of the Catalan generating function and explain its significance regarding the growth of C_n .

Solutions

1. Use the binomial expansion of $(1 - 4x)^{1/2}$ to find the coefficient of x^3 in $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.

- Expand $(1 - 4x)^{1/2}$ using the binomial theorem: $1 + \frac{1}{2}(-4x) + \frac{(1/2)(-1/2)}{2}(-4x)^2 + \frac{(1/2)(-1/2)(-3/2)}{6}(-4x)^3 + \frac{(1/2)(-1/2)(-3/2)(-5/2)}{24}(-4x)^4 + \dots$
- Simplify terms: $1 - 2x - 2x^2 - 4x^3 - 10x^4 - \dots$
- Substitute into $C(x)$: $\frac{1 - (1 - 2x - 2x^2 - 4x^3 - 10x^4 - \dots)}{2x} = \frac{2x + 2x^2 + 4x^3 + 10x^4 + \dots}{2x}$
- Divide by $2x$: $1 + 1x + 2x^2 + 5x^3 + \dots$
- The coefficient of x^3 is 5.

Answer: 5

2. Show that the function $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ satisfies the equation $C(x) = 1 + xC(x)^2$.

- Start with $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.
- Compute $C(x)^2 = \frac{(1 - \sqrt{1 - 4x})^2}{4x^2} = \frac{1 + (1 - 4x) - 2\sqrt{1 - 4x}}{4x^2} = \frac{2 - 4x - 2\sqrt{1 - 4x}}{4x^2}$.
- Multiply by x : $xC(x)^2 = \frac{2 - 4x - 2\sqrt{1 - 4x}}{4x} = \frac{1 - 2x - \sqrt{1 - 4x}}{2x}$.
- Add 1: $1 + xC(x)^2 = \frac{2x + 1 - 2x - \sqrt{1 - 4x}}{2x} = \frac{1 - \sqrt{1 - 4x}}{2x}$.
- This is exactly $C(x)$.

Answer: The identity holds.

3. Determine the radius of convergence of the Catalan generating function and explain its significance regarding the growth of C_n .

- The function $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ has a singularity where the square root argument is zero: $1 - 4x = 0$.
- Solving for x gives $x = 1/4$.
- The radius of convergence is $R = 1/4$.
- By the root test, $\limsup_{n \rightarrow \infty} (C_n)^{1/n} = 1/R = 4$.
- This implies that the Catalan numbers grow roughly as 4^n .

Answer: $R = 1/4$; the sequence grows exponentially as 4^n .