

Generating functions III — counting with products

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Practice problems

1. Find the number of ways to make a total of 7 using only 1p and 3p coins.
2. How many ways can you choose a total of 5 items from two categories, A and B, if you must choose at least one of A and at most two of B?
3. Find the number of ways to make a total of 10 using 1p, 2p, and 5p coins, but you are only allowed to use at most one 5p coin.

Solutions

1. Find the number of ways to make a total of 7 using only 1p and 3p coins.

- The generating function for 1p coins is $G_1(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$.
- The generating function for 3p coins is $G_3(x) = 1 + x^3 + x^6 + \dots = \frac{1}{1-x^3}$.
- The total generating function is $G(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \dots)(1 + x^3 + x^6 + \dots)$.
- To get x^7 , we can have $x^7 \cdot 1$, $x^4 \cdot x^3$, or $x^1 \cdot x^6$.
- There are 3 such pairs.

Answer: 3

2. How many ways can you choose a total of 5 items from two categories, A and B, if you must choose at least one of A and at most two of B?

- For A, the generating function is $G_A(x) = x + x^2 + x^3 + x^4 + x^5 + \dots = \frac{x}{1-x}$.
- For B, the generating function is $G_B(x) = 1 + x + x^2$.
- We want the coefficient of x^5 in $G(x) = (x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2)$.
- The pairs (a, b) that sum to 5 are $(5, 0)$, $(4, 1)$, $(3, 2)$.
- There are 3 ways.

Answer: 3

3. Find the number of ways to make a total of 10 using 1p, 2p, and 5p coins, but you are only allowed to use at most one 5p coin.

- The generating function for 1p is $\frac{1}{1-x}$.
- The generating function for 2p is $\frac{1}{1-x^2}$.
- The generating function for 5p is $(1 + x^5)$ because we can use zero or one.
- Total $G(x) = \frac{1+x^5}{(1-x)(1-x^2)}$.
- Case 1: Zero 5p coins. We need 10 from 1p and 2p. The ways are $(10, 0)$, $(8, 1)$, $(6, 2)$, $(4, 3)$, $(2, 4)$, $(0, 5)$, which is 6 ways.
- Case 2: One 5p coin. We need 5 from 1p and 2p. The ways are $(5, 0)$, $(3, 1)$, $(1, 2)$, which is 3 ways.
- Total ways = $6 + 3 = 9$.

Answer: 9