

Generating functions II — solving recurrences by algebra

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Practice problems

1. Solve the recurrence $a_n = 4a_{n-1}$ for $n \geq 1$ with $a_0 = 5$ using generating functions.
2. Solve the recurrence $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$ using generating functions.
3. Find the closed form for $a_n = 2a_{n-1} + 1$ for $n \geq 1$ with $a_0 = 0$ using generating functions.

Solutions

1. Solve the recurrence $a_n = 4a_{n-1}$ for $n \geq 1$ with $a_0 = 5$ using generating functions.

- Define $G(x) = \sum_{n=0}^{\infty} a_n x^n$.
- Multiply recurrence by x^n and sum from $n = 1$: $\sum_{n=1}^{\infty} a_n x^n = 4 \sum_{n=1}^{\infty} a_{n-1} x^n$.
- Rewrite in terms of $G(x)$: $G(x) - a_0 = 4xG(x)$.
- Substitute $a_0 = 5$: $G(x) - 5 = 4xG(x)$.
- Solve for $G(x)$: $G(x)(1 - 4x) = 5 \implies G(x) = \frac{5}{1-4x}$.
- Decode using the geometric series: $a_n = 5 \cdot 4^n$.

Answer: $a_n = 5 \cdot 4^n$

2. Solve the recurrence $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$ using generating functions.

- Sum from $n = 2$: $\sum_{n=2}^{\infty} a_n x^n = 3 \sum_{n=2}^{\infty} a_{n-1} x^n - 2 \sum_{n=2}^{\infty} a_{n-2} x^n$.
- Substitute $G(x)$: $G(x) - a_0 - a_1 x = 3x(G(x) - a_0) - 2x^2 G(x)$.
- Substitute $a_0 = 0, a_1 = 1$: $G(x) - x = 3xG(x) - 2x^2 G(x)$.
- Solve for $G(x)$: $G(x)(1 - 3x + 2x^2) = x \implies G(x) = \frac{x}{(1-x)(1-2x)}$.
- Partial fractions: $\frac{x}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{1}{1-2x}$.
- Decode: $a_n = -1(1^n) + 1(2^n) = 2^n - 1$.

Answer: $a_n = 2^n - 1$

3. Find the closed form for $a_n = 2a_{n-1} + 1$ for $n \geq 1$ with $a_0 = 0$ using generating functions.

- Sum from $n = 1$: $\sum_{n=1}^{\infty} a_n x^n = 2 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n$.
- Substitute $G(x)$: $G(x) - a_0 = 2xG(x) + \frac{x}{1-x}$.
- Substitute $a_0 = 0$: $G(x)(1 - 2x) = \frac{x}{1-x}$.
- Solve for $G(x)$: $G(x) = \frac{x}{(1-x)(1-2x)}$.
- Partial fractions: $G(x) = \frac{-1}{1-x} + \frac{1}{1-2x}$.
- Decode: $a_n = 2^n - 1$.

Answer: $a_n = 2^n - 1$