Generating functions I — the sequence-to-function dictionary

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Practice problems

1. Find the ordinary generating function for the sequence $a_n = 3^n$ for $n \geq 0$.
2. Find the sequence a_n corresponding to the generating function $G(x) = \frac{x^3}{1-x}$.
3. Find the generating function for the sequence $a_n = n \cdot 2^n$.

Solutions

1. Find the ordinary generating function for the sequence $a_n = 3^n$ for $n \geq 0$.

   - The sequence is $1, 3, 9, 27, \ldots$
   - The generating function is $G(x) = \sum_{n=0}^{\infty} 3^n x^n$.
   - This is a geometric series with common ratio $3x$.
   - Using the formula $\frac{1}{1-r}$, we substitute $r = 3x$.

   Answer: G(x) = \frac11-3x

2. Find the sequence a_n corresponding to the generating function $G(x) = \frac{x^3}{1-x}$.

   - We know that $\frac{1}{1-x}$ corresponds to the sequence $1, 1, 1, \ldots$ (all ones).
   - Multiplying by x^3 shifts the sequence to the right by 3 positions.
   - The first three terms a_0, a_1, a_2 become 0.
   - The terms from a_3 onwards are all 1.

   Answer: a_n = 0 \text for n ¡ 3, a_n = 1 \text for n \ge 3

3. Find the generating function for the sequence $a_n = n \cdot 2^n$.

   - We know the generating function for $b_n = 2^n$ is $B(x) = \frac{1}{1-2x}$.
   - To introduce the factor of n, we use the operator $x\frac{d}{dx}$.
   - Compute the derivative: $\frac{d}{dx}(1-2x)^{-1} = -1(1-2x)^{-2} \cdot (-2) = \frac{2}{(1-2x)^2}$.
   - Multiply by x to get $G(x) = \frac{2x}{(1-2x)^2}$.

   Answer: G(x) = \frac2x(1-2x)^2