

Nonhomogeneous Recurrences

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Practice problems

1. Solve the recurrence $a_n = 3a_{n-1} + 4$ with base case $a_0 = 2$.
2. Solve the recurrence $a_n = 2a_{n-1} + n$ with base case $a_0 = 1$.
3. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ with $a_0 = 1, a_1 = 4$.

Solutions

1. Solve the recurrence $a_n = 3a_{n-1} + 4$ with base case $a_0 = 2$.

- Find homogeneous solution: $r - 3 = 0 \implies a_n^{(h)} = A3^n$.
- Guess particular solution: $a_n^{(p)} = K$. Plug in: $K = 3K + 4 \implies -2K = 4 \implies K = -2$.
- General solution: $a_n = A3^n - 2$.
- Use base case $a_0 = 2$: $A3^0 - 2 = 2 \implies A - 2 = 2 \implies A = 4$.
- Final solution: $a_n = 4 \times 3^n - 2$.

Answer: $a_n = 4 \times 3^n - 2$

2. Solve the recurrence $a_n = 2a_{n-1} + n$ with base case $a_0 = 1$.

- Homogeneous solution: $r - 2 = 0 \implies a_n^{(h)} = A2^n$.
- Guess particular solution: $a_n^{(p)} = Kn + L$.
- Plug in: $Kn + L = 2(K(n-1) + L) + n = 2Kn - 2K + 2L + n$.
- Match coefficients: $K = 2K + 1 \implies K = -1$. Constant terms: $L = -2K + 2L \implies L = 2K = -2$.
- General solution: $a_n = A2^n - n - 2$.
- Use base case $a_0 = 1$: $A - 0 - 2 = 1 \implies A = 3$.
- Final solution: $a_n = 3 \times 2^n - n - 2$.

Answer: $a_n = 3 \times 2^n - n - 2$

3. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ with $a_0 = 1, a_1 = 4$.

- Homogeneous: $r^2 - 4r + 4 = 0 \implies (r - 2)^2 = 0$. Roots are 2, 2. $a_n^{(h)} = A2^n + Bn2^n$.
- Forcing function 2^n is a double root of the characteristic equation. Guess $a_n^{(p)} = Kn^22^n$.
- Plug in: $Kn^22^n = 4K(n-1)^22^{n-1} - 4K(n-2)^22^{n-2} + 2^n$.
- Divide by 2^{n-2} : $4Kn^2 = 8K(n^2 - 2n + 1) - 4K(n^2 - 4n + 4) + 4$.
- Simplify: $4Kn^2 = 8Kn^2 - 16Kn + 8K - 4Kn^2 + 16Kn - 16K + 4$.
- The n^2 and n terms cancel: $0 = -8K + 4 \implies K = 0.5$.
- General solution: $a_n = A2^n + Bn2^n + 0.5n^22^n$.
- Base cases: $a_0 = 1 \implies A = 1$. $a_1 = 4 \implies 2A + 2B + 0.5(1)(2) = 4 \implies 2 + 2B + 1 = 4 \implies 2B = 1 \implies B = 0.5$.
- Final solution: $a_n = 2^n + 0.5n2^n + 0.5n^22^n = (1 + 0.5n + 0.5n^2)2^n$.

Answer: $a_n = (1 + 0.5n + 0.5n^2)2^n$