

Solving linear recurrences II — repeated and complex roots

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Practice problems

1. Find the closed form for the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ with base cases $a_0 = 1$ and $a_1 = 6$.
2. Solve the recurrence $a_n = 4a_{n-1} - 8a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$.
3. A sequence is defined by $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$. Find the closed form.

Solutions

1. Find the closed form for the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ with base cases $a_0 = 1$ and $a_1 = 6$.

- The characteristic equation is $r^2 - 6r + 9 = 0$, which factors as $(r - 3)^2 = 0$.
- The root is $r = 3$ (repeated), so the general solution is $a_n = A(3^n) + Bn(3^n)$.
- Using $a_0 = 1$: $A(3^0) + B(0)(3^0) = 1$, so $A = 1$.
- Using $a_1 = 6$: $1(3^1) + B(1)(3^1) = 6$, which means $3 + 3B = 6$, so $3B = 3$ and $B = 1$.
- The closed form is $a_n = 3^n + n3^n = (n + 1)3^n$.

Answer: $a_n = (n+1)3^n$

2. Solve the recurrence $a_n = 4a_{n-1} - 8a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$.

- The characteristic equation is $r^2 - 4r + 8 = 0$.
- Using the quadratic formula: $r = \frac{4 \pm \sqrt{16 - 32}}{2} = 2 \pm 2i$.
- Magnitude $\rho = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$. Angle $\theta = \text{atan}(2/2) = \pi/4$.
- General solution: $a_n = (2\sqrt{2})^n (C \cos(n\pi/4) + D \sin(n\pi/4))$.
- For $a_0 = 1$: $1(C(1) + D(0)) = 1$, so $C = 1$.
- For $a_1 = 2$: $2\sqrt{2}(1 \cdot \frac{1}{\sqrt{2}} + D \cdot \frac{1}{\sqrt{2}}) = 2$, which simplifies to $2 + 2D = 2$, so $D = 0$.
- The closed form is $a_n = (2\sqrt{2})^n \cos(n\pi/4)$.

Answer: $a_n = (2\sqrt{2})^n \cos(n\pi/4)$

3. A sequence is defined by $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$. Find the closed form.

- The characteristic equation is $r^2 - 2r + 1 = 0$, which is $(r - 1)^2 = 0$.
- The root is $r = 1$ (repeated), so $a_n = A(1^n) + Bn(1^n) = A + Bn$.
- Using $a_0 = 0$: $A + B(0) = 0$, so $A = 0$.
- Using $a_1 = 1$: $0 + B(1) = 1$, so $B = 1$.
- The closed form is $a_n = n$.

Answer: $a_n = n$