

Solving linear recurrences I — the characteristic equation

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Practice problems

1. Find the closed form for the recurrence $a_n = 4a_{n-1} - 3a_{n-2}$ with $a_0 = 2$ and $a_1 = 4$.
2. Solve the recurrence $a_n = a_{n-1} + 6a_{n-2}$ with $a_0 = 5$ and $a_1 = 5$.
3. A sequence is defined by $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$. If $a_0 = 2$ and $a_1 = 7$, find the formula for a_n .

Solutions

1. Find the closed form for the recurrence $a_n = 4a_{n-1} - 3a_{n-2}$ with $a_0 = 2$ and $a_1 = 4$.

- The characteristic equation is $r^2 - 4r + 3 = 0$.
- Factoring gives $(r - 1)(r - 3) = 0$, so roots are $r_1 = 1$ and $r_2 = 3$.
- General solution: $a_n = A(1^n) + B(3^n) = A + B(3^n)$.
- Using $a_0 = 2$: $A + B = 2$.
- Using $a_1 = 4$: $A + 3B = 4$.
- Subtracting the first from the second: $2B = 2$, so $B = 1$. Then $A = 1$.

Answer: $a_n = 1 + 3^n$

2. Solve the recurrence $a_n = a_{n-1} + 6a_{n-2}$ with $a_0 = 5$ and $a_1 = 5$.

- Characteristic equation: $r^2 - r - 6 = 0$.
- Factoring: $(r - 3)(r + 2) = 0$, so roots are 3 and -2 .
- General solution: $a_n = A(3^n) + B(-2^n)$.
- Base cases: $A + B = 5$ and $3A - 2B = 5$.
- From first: $B = 5 - A$. Substitute: $3A - 2(5 - A) = 5 \implies 5A - 10 = 5 \implies 5A = 15 \implies A = 3$.
- Then $B = 5 - 3 = 2$.

Answer: $a_n = 3(3^n) + 2(-2^n)$

3. A sequence is defined by $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$. If $a_0 = 2$ and $a_1 = 7$, find the formula for a_n .

- Characteristic equation: $r^2 - 7r + 10 = 0$.
- Factoring: $(r - 2)(r - 5) = 0$, so roots are 2 and 5.
- General solution: $a_n = A(2^n) + B(5^n)$.
- Base cases: $A + B = 2$ and $2A + 5B = 7$.
- Multiply first by 2: $2A + 2B = 4$.
- Subtract from second: $3B = 3 \implies B = 1$.
- Then $A = 2 - 1 = 1$.

Answer: $a_n = 2^n + 5^n$