

Recurrences everywhere: rabbits, stairs, towers

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Practice problems

1. A person climbs a staircase with n steps. They can take 1, 2, or 3 steps at a time. Let a_n be the number of ways to reach the n -th step. Find the recurrence relation and the number of ways to climb 4 steps, given $a_0 = 1, a_1 = 1, a_2 = 2$.
2. A string of length n is made of characters 'A' and 'B'. No two 'B's can be adjacent. Let a_n be the number of such strings. Find the recurrence relation and the value of a_5 .
3. In a modified Tower of Hanoi, you have n discs, but you can only move a disc to a peg if that peg is empty or contains a disc at least two sizes larger. However, for the standard version we studied, if a_n is the minimum moves for n discs, prove by induction or computation that $a_n = 2^n - 1$ for $n = 1, 2, 3$.

Solutions

1. A person climbs a staircase with n steps. They can take 1, 2, or 3 steps at a time. Let a_n be the number of ways to reach the n -th step. Find the recurrence relation and the number of ways to climb 4 steps, given $a_0 = 1, a_1 = 1, a_2 = 2$.

- The last move to reach step n must be a 1-step, a 2-step, or a 3-step.
- This gives the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.
- For $n = 3$: $a_3 = a_2 + a_1 + a_0 = 2 + 1 + 1 = 4$.
- For $n = 4$: $a_4 = a_3 + a_2 + a_1 = 4 + 2 + 1 = 7$.

Answer: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, $a_4 = 7$

2. A string of length n is made of characters 'A' and 'B'. No two 'B's can be adjacent. Let a_n be the number of such strings. Find the recurrence relation and the value of a_5 .

- If the string ends in 'A', the previous $n - 1$ characters can be any valid string: a_{n-1} ways.
- If the string ends in 'B', the character before it must be 'A'. The previous $n - 2$ characters can be any valid string: a_{n-2} ways.
- The recurrence is $a_n = a_{n-1} + a_{n-2}$.
- Base cases: $a_1 = 2$ ('A', 'B'), $a_2 = 3$ ('AA', 'AB', 'BA').
- Compute: $a_3 = 3 + 2 = 5, a_4 = 5 + 3 = 8, a_5 = 8 + 5 = 13$.

Answer: $a_n = a_{n-1} + a_{n-2}$, $a_5 = 13$

3. In a modified Tower of Hanoi, you have n discs, but you can only move a disc to a peg if that peg is empty or contains a disc at least two sizes larger. However, for the standard version we studied, if a_n is the minimum moves for n discs, prove by induction or computation that $a_n = 2^n - 1$ for $n = 1, 2, 3$.

- For $n = 1$: $a_1 = 2^1 - 1 = 1$. Matches base case.
- For $n = 2$: $a_2 = 2(1) + 1 = 3$. Also $2^2 - 1 = 3$.
- For $n = 3$: $a_3 = 2(3) + 1 = 7$. Also $2^3 - 1 = 7$.

Answer: $a_1=1, a_2=3, a_3=7$