

The Probabilistic Method — a Taste

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Practice problems

1. Use the probabilistic method to determine if it proves $R(3, 3) > 2$.
2. Using the probabilistic method, what is the largest n for which we can prove $R(4, 4) > n$ using the formula $\binom{n}{4} 2^{1-\binom{4}{2}} < 1$?
3. Prove that $R(5, 5) > 10$ using the probabilistic method.

Solutions

1. Use the probabilistic method to determine if it proves $R(3, 3) > 2$.

- Identify $n = 2$ and $r = 3$.
- Calculate the number of subsets of size 3: $\binom{2}{3} = 0$.
- Calculate the probability of a monochromatic triangle: $0 \cdot 2^{1-3} = 0$.
- Since $0 < 1$, the condition is satisfied.

Answer: Yes, it proves $R(3, 3) > 2$.

2. Using the probabilistic method, what is the largest n for which we can prove $R(4, 4) > n$ using the formula $\binom{n}{4} 2^{1-\binom{4}{2}} < 1$?

- The probability for a fixed set of 4 is $2^{1-6} = 1/32$.
- We need $\binom{n}{4} < 32$.
- For $n = 6$, $\binom{6}{4} = 15 < 32$.
- For $n = 7$, $\binom{7}{4} = 35 \not< 32$.

Answer: $n = 6$

3. Prove that $R(5, 5) > 10$ using the probabilistic method.

- Set $n = 10$ and $r = 5$.
- Calculate $\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252$.
- Calculate the probability of a monochromatic K_5 : $2^{1-\binom{5}{2}} = 2^{1-10} = 2^{-9} = 1/512$.
- Check the inequality: $252 \cdot (1/512) = 252/512 \approx 0.492$.
- Since $0.492 < 1$, the probabilistic method proves $R(5, 5) > 10$.

Answer: The probability of failure is $252/512 < 1$, so $R(5, 5) > 10$.