

Ramsey theory — complete disorder is impossible

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Practice problems

1. Explain why $R(3,3) > 5$ by describing a specific edge-colouring of a complete graph with 5 vertices that contains no monochromatic triangle.
2. In the proof of $R(3,3) = 6$, we use the Pigeonhole Principle on the edges incident to a single vertex. If the vertex has degree 5 and there are 2 colours, what is the minimum number of edges of the same colour?
3. Suppose we have a group of 6 people. If we know that person A has exactly 3 red edges and 2 blue edges, and we find that the 3 people connected by red edges are all mutual strangers, have we found a monochromatic triangle? If so, which one?

Solutions

1. Explain why $R(3, 3) > 5$ by describing a specific edge-colouring of a complete graph with 5 vertices that contains no monochromatic triangle.

- Arrange 5 vertices in a circle.
- Colour the edges of the outer pentagon red.
- Colour the edges of the inner star (the diagonals) blue.
- Check all possible triangles: any three vertices will have at least one red edge and at least one blue edge.
- Since no monochromatic triangle exists, $R(3, 3)$ must be greater than 5.

Answer: A pentagon of red edges and a star of blue edges avoids monochromatic triangles.

2. In the proof of $R(3, 3) = 6$, we use the Pigeonhole Principle on the edges incident to a single vertex. If the vertex has degree 5 and there are 2 colours, what is the minimum number of edges of the same colour?

- The number of items is $n = 5$ (edges).
- The number of containers is $k = 2$ (colours).
- By the Pigeonhole Principle, at least $\lceil n/k \rceil$ items must be in one container.
- $\lceil 5/2 \rceil = 3$.

Answer: 3

3. Suppose we have a group of 6 people. If we know that person A has exactly 3 red edges and 2 blue edges, and we find that the 3 people connected by red edges are all mutual strangers, have we found a monochromatic triangle? If so, which one?

- Person A is connected to B, C, and D via red edges.
- The problem states B, C, and D are mutual strangers.
- In our graph model, 'mutual strangers' means the edges between B, C, and D are all blue.
- Therefore, the vertices B, C, and D form a blue triangle.
- Yes, a monochromatic triangle exists.

Answer: Yes, a blue triangle formed by the three people who are mutual strangers.