Burnside's Lemma — Counting Necklaces Under Symmetry

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Practice problems

1. How many distinct necklaces can be made with 5 beads and 3 colours, considering only rotations?
2. A square tile is coloured with 2 colours on its 4 corners. How many distinct tiles are there under rotation only?
3. How many distinct necklaces with 6 beads and 3 colours exist under rotation only?

Solutions

1. How many distinct necklaces can be made with 5 beads and 3 colours, considering only rotations?

   - The group size is $|G| = 5$.
   - For $k = 0$, $\gcd(5, 0) = 5$, so $|X^{r_0}| = 3^5 = 243$.
   - For $k = 1, 2, 3, 4$, $\gcd(5, k) = 1$, so $|X^{r_k}| = 3^1 = 3$.
   - Sum of fixed points: $243 + 3 + 3 + 3 + 3 = 255$.
   - Number of orbits: $255/5 = 51$.

   Answer: 51

2. A square tile is coloured with 2 colours on its 4 corners. How many distinct tiles are there under rotation only?

   - The group size is $|G| = 4$.
   - For $k = 0$, $\gcd(4, 0) = 4$, so $|X^{r_0}| = 2^4 = 16$.
   - For $k = 1$, $\gcd(4, 1) = 1$, so $|X^{r_1}| = 2^1 = 2$.
   - For $k = 2$, $\gcd(4, 2) = 2$, so $|X^{r_2}| = 2^2 = 4$.
   - For $k = 3$, $\gcd(4, 3) = 1$, so $|X^{r_3}| = 2^1 = 2$.
   - Sum: $16 + 2 + 4 + 2 = 24$.
   - Orbits: $24/4 = 6$.

   Answer: 6

3. How many distinct necklaces with 6 beads and 3 colours exist under rotation only?

   - The group size is $|G| = 6$.
   - For $k = 0$, $\gcd(6, 0) = 6$, so $|X^{r_0}| = 3^6 = 729$.
   - For $k = 1$, $\gcd(6, 1) = 1$, so $|X^{r_1}| = 3^1 = 3$.
   - For $k = 2$, $\gcd(6, 2) = 2$, so $|X^{r_2}| = 3^2 = 9$.
   - For $k = 3$, $\gcd(6, 3) = 3$, so $|X^{r_3}| = 3^3 = 27$.
   - For $k = 4$, $\gcd(6, 4) = 2$, so $|X^{r_4}| = 3^2 = 9$.
   - For $k = 5$, $\gcd(6, 5) = 1$, so $|X^{r_5}| = 3^1 = 3$.
   - Sum: $729 + 3 + 9 + 27 + 9 + 3 = 780$.
   - Orbits: $780/6 = 130$.

   Answer: 130
