The Catalan numbers II — the reflection proof

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Practice problems

1. Compute the 5th Catalan number C_5 using the formula $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.
2. If we were counting paths from $(0,0)$ to (n,n) that never touch the line $y = x + 2$, what would be the destination of the reflected bad paths?
3. Show that the number of paths from $(0,0)$ to (n,n) that stay below the diagonal is equal to the number of ways to arrange n pairs of balanced parentheses. Explain the mapping.

Solutions

1. Compute the 5th Catalan number C_5 using the formula $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

   - Identify $n = 5$, so we need $\binom{10}{5} - \binom{10}{4}$.
   - Calculate $\binom{10}{5} = \frac{10\cdot9\cdot8\cdot7\cdot6}{5\cdot4\cdot3\cdot2\cdot1} = 252$.
   - Calculate $\binom{10}{4} = \frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1} = 210$.
   - Subtract the results: $252 - 210 = 42$.

   Answer: 42

2. If we were counting paths from $(0, 0)$ to (n, n) that never touch the line $y = x + 2$, what would be the destination of the reflected bad paths?

   - The reflection line is $y = x + 2$.
   - The reflection of a point (x, y) across $y = x + k$ is $(y - k, x + k)$.
   - Here $k = 2$, so the point (n, n) reflects to $(n - 2, n + 2)$.

   Answer: (n-2, n+2)

3. Show that the number of paths from $(0, 0)$ to (n, n) that stay below the diagonal is equal to the number of ways to arrange n pairs of balanced parentheses. Explain the mapping.

   - Map each 'Right' step to an opening parenthesis '(' and each 'Up' step to a closing parenthesis ')'.
   - A path stays below the diagonal if at any point, the number of Right steps is greater than or equal to the number of Up steps.
   - In parentheses, this means at any point, the number of '(' is greater than or equal to the number of ')'.
   - Since the path ends at (n, n), there are exactly n of each, resulting in a balanced string of $2n$ parentheses.

   Answer: The mapping is Right $\to$ '(' and Up $\to$ ')'.
