

The Catalan numbers I — one sequence, a dozen disguises

Lesson 40 · Module 3 — Combinatorics, Deepened · Discrete Mathematics · dailymaths.uk/discrete

Practice problems

1. How many ways can you arrange 3 pairs of balanced parentheses?
2. A convex polygon has 6 sides. How many ways can it be divided into triangles using non-intersecting diagonals?
3. How many paths from $(0, 0)$ to $(5, 5)$ stay on or below the line $y = x$?

Solutions

1. How many ways can you arrange 3 pairs of balanced parentheses?

- Identify that $n = 3$ pairs of parentheses corresponds to the 3rd Catalan number C_3 .
- Apply the formula $C_n = \frac{1}{n+1} \binom{2n}{n}$.
- Compute $\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$.
- Calculate $C_3 = \frac{1}{4} \times 20 = 5$.

Answer: 5

2. A convex polygon has 6 sides. How many ways can it be divided into triangles using non-intersecting diagonals?

- The number of triangulations of a polygon with k sides is C_{k-2} .
- Here $k = 6$, so we need $C_{6-2} = C_4$.
- Apply the formula $C_4 = \frac{1}{5} \binom{8}{4}$.
- Compute $\binom{8}{4} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$.
- Calculate $C_4 = \frac{70}{5} = 14$.

Answer: 14

3. How many paths from $(0,0)$ to $(5,5)$ stay on or below the line $y = x$?

- This is the definition of the 5th Catalan number C_5 .
- Apply the formula $C_5 = \frac{1}{6} \binom{10}{5}$.
- Compute $\binom{10}{5} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$.
- Calculate $C_5 = \frac{252}{6} = 42$.

Answer: 42