Integer partitions; Ferrers diagrams and conjugate partitions

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Practice problems

1. Find the conjugate partition of $12 = 6 + 3 + 2 + 1$.
2. List all partitions of 6 that have exactly 3 parts.
3. Using the conjugation principle, how many partitions of 7 have a largest part equal to 2?

Solutions

1. Find the conjugate partition of $12 = 6 + 3 + 2 + 1$.

   - Draw the Ferrers diagram: Row 1 has 6 dots, Row 2 has 3, Row 3 has 2, Row 4 has 1.
   - Count the columns: Column 1 has 4 dots, Column 2 has 3, Column 3 has 3, Column 4 has 1, Column 5 has 1, Column 6 has 1.
   - The conjugate partition is the sequence of column lengths.

   Answer: 4 + 3 + 3 + 1 + 1 + 1

2. List all partitions of 6 that have exactly 3 parts.

   - We need $n_1 + n_2 + n_3 = 6$ with $n_1 \geq n_2 \geq n_3 \geq 1$.
   - If $n_1 = 4$, then $n_2 + n_3 = 2$, which forces $n_2 = 1, n_3 = 1$. Partition: $4 + 1 + 1$.
   - If $n_1 = 3$, then $n_2 + n_3 = 3$, which forces $n_2 = 2, n_3 = 1$. Partition: $3 + 2 + 1$.
   - If $n_1 = 2$, then $n_2 + n_3 = 4$, which forces $n_2 = 2, n_3 = 2$. Partition: $2 + 2 + 2$.

   Answer: 4+1+1, 3+2+1, 2+2+2

3. Using the conjugation principle, how many partitions of 7 have a largest part equal to 2?

   - The number of partitions of 7 with largest part 2 is equal to the number of partitions of 7 into exactly 2 parts.
   - We list partitions of 7 into exactly 2 parts: $6 + 1, 5 + 2, 4 + 3$.
   - There are 3 such partitions.

   Answer: 3
