Combinatorial proof II — binomial identity clinic

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Practice problems

1. Use the Hockey-stick identity to compute the value of $\sum_{i=3}^{10} \binom{i}{3}$.
2. A committee of 5 people is to be chosen from a group of 7 men and 6 women. Use Vandermonde's identity to express the total number of ways as a sum of products of binomial coefficients.
3. Prove the identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ using a combinatorial argument.

Solutions

1. Use the Hockey-stick identity to compute the value of $\sum_{i=3}^{10} \binom{i}{3}$.

   - Identify $k = 3$ and $n = 10$ in the identity $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$.
   - Substitute the values: $\sum_{i=3}^{10} \binom{i}{3} = \binom{10+1}{3+1} = \binom{11}{4}$.
   - Calculate $\binom{11}{4} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$.

   Answer: 330

2. A committee of 5 people is to be chosen from a group of 7 men and 6 women. Use Vandermonde's identity to express the total number of ways as a sum of products of binomial coefficients.

   - The total number of ways is $\binom{7+6}{5} = \binom{13}{5}$.
   - Using Vandermonde's identity $\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$ with $n = 7, m = 6, r = 5$.
   - The sum is $\sum_{k=0}^{5} \binom{7}{k}\binom{6}{5-k} = \binom{7}{0}\binom{6}{5} + \binom{7}{1}\binom{6}{4} + \binom{7}{2}\binom{6}{3} + \binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} + \binom{7}{5}\binom{6}{0}$.

   Answer: \sum_k=0ˆ5 \binom7k \binom65-k

3. Prove the identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ using a combinatorial argument.

   - Consider a set of $2n$ objects divided into two groups of n objects each.
   - The number of ways to choose n objects from the $2n$ is $\binom{2n}{n}$.
   - Alternatively, we can choose k objects from the first group and $n - k$ objects from the second group for $k = 0, 1, \ldots, n$.
   - The number of ways to do this is $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.
   - Since $\binom{n}{n-k} = \binom{n}{k}$, the sum becomes $\sum_{k=0}^{n} \binom{n}{k}^2$.

   Answer: \binom2nn
