Combinatorial proof I — counting one thing two ways

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Practice problems

1. Prove the identity $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$ using a combinatorial proof. Hint: Imagine picking a committee of size k and then picking a sub-committee of size m from within that committee.
2. Use a combinatorial argument to show that $\binom{n}{k} = \binom{n}{n-k}$.
3. Prove that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ using a combinatorial proof.

Solutions

1. Prove the identity $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$ using a combinatorial proof. Hint: Imagine picking a committee of size k and then picking a sub-committee of size m from within that committee.

   - Step 1: Define the set. We are counting the number of ways to choose a committee of k people from n, and then choosing m of those k people to be on a special sub-committee.
   - Step 2: Count method 1. First choose the k committee members in $\binom{n}{k}$ ways, then choose m sub-committee members from the k in $\binom{k}{m}$ ways. Total: $\binom{n}{k}\binom{k}{m}$.
   - Step 3: Count method 2. First choose the m sub-committee members from the total n in $\binom{n}{m}$ ways. Then, choose the remaining $k-m$ committee members from the remaining $n-m$ people in $\binom{n-m}{k-m}$ ways. Total: $\binom{n}{m}\binom{n-m}{k-m}$.
   - Step 4: Since both methods count the same set of configurations, the expressions must be equal.

   Answer: \binomnk \binomkm = \binomnm \binomn-mk-m

2. Use a combinatorial argument to show that $\binom{n}{k} = \binom{n}{n-k}$.

   - Step 1: Define the set. We are counting the number of ways to choose a subset of k elements from a set of n elements.
   - Step 2: Method 1. By definition, there are $\binom{n}{k}$ ways to choose k elements to be in the subset.
   - Step 3: Method 2. Choosing k elements to be in the subset is logically identical to choosing $n-k$ elements to be excluded from the subset. There are $\binom{n}{n-k}$ ways to choose the excluded elements.
   - Step 4: Because every selection of k elements uniquely determines a selection of $n-k$ elements, the counts are equal.

   Answer: \binomnk = \binomnn-k

3. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ using a combinatorial proof.

   - Step 1: Define the set. We are counting the total number of subsets of a set with n elements (the power set).
   - Step 2: Method 1. We can count subsets by their size k. For a fixed k, there are $\binom{n}{k}$ subsets. Summing from $k = 0$ to n gives $\sum_{k=0}^{n} \binom{n}{k}$.
   - Step 3: Method 2. For each of the n elements, we have a binary choice: it is either in the subset or it is not. By the multiplication principle, there are $2 \times 2 \times \cdots \times 2$ (n times), which is 2^n ways.
   - Step 4: Both methods count the total number of subsets, so the identity holds.

   Answer: \sum_k=0ˆn \binomnk = 2ˆn