Primes; the Fundamental Theorem of Arithmetic — proved

Lesson 24 · Module 2 — Number Theory & Cryptography · Discrete Mathematics · daily-maths.uk/discrete

Practice problems

1. Find the canonical prime factorization of 1260.
2. Use prime factorization to find the greatest common divisor of 420 and 546.
3. Prove that if n is an integer such that $n > 1$ and n is not divisible by any prime $p \leq \sqrt{n}$, then n must be prime.

Solutions

1. Find the canonical prime factorization of 1260.

   - Divide by 2: $1260 = 2 \times 630$
   - Divide by 2 again: $630 = 2 \times 315$, so $1260 = 2^2 \times 315$
   - Divide by 3: $315 = 3 \times 105$, so $1260 = 2^2 \times 3 \times 105$
   - Divide by 3 again: $105 = 3 \times 35$, so $1260 = 2^2 \times 3^2 \times 35$
   - Divide by 5: $35 = 5 \times 7$, so $1260 = 2^2 \times 3^2 \times 5 \times 7$

   Answer: 2ˆ2 \times 3ˆ2 \times 5ˆ1 \times 7ˆ1

2. Use prime factorization to find the greatest common divisor of 420 and 546.

   - Factor 420: $420 = 2 \times 210 = 2^2 \times 105 = 2^2 \times 3 \times 35 = 2^2 \times 3 \times 5 \times 7$
   - Factor 546: $546 = 2 \times 273 = 2 \times 3 \times 91 = 2 \times 3 \times 7 \times 13$
   - Identify common primes with minimum exponents: 2^1, 3^1, and 7^1
   - Compute GCD: $2 \times 3 \times 7 = 42$

   Answer: 42

3. Prove that if n is an integer such that $n > 1$ and n is not divisible by any prime $p \leq \sqrt{n}$, then n must be prime.

   - Assume for contradiction that n is composite.
   - Then $n = a \times b$ for some integers $1 < a \leq b < n$.
   - If $a > \sqrt{n}$, then $b \geq a > \sqrt{n}$, which implies $a \times b > \sqrt{n} \times \sqrt{n} = n$.
   - This contradicts $a \times b = n$, so we must have $a \leq \sqrt{n}$.
   - Since $a > 1$, a must have at least one prime factor p.
   - Since $p \leq a \leq \sqrt{n}$, n is divisible by a prime $p \leq \sqrt{n}$, which contradicts the hypothesis.

   Answer: n is prime
