Divisibility and the division algorithm

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Practice problems

1. Use the division algorithm to find the unique integers q and r such that $1024 = q(37) + r$, where $0 \leq r < 37$.
2. Prove that if $d \mid a$ and $d \mid b$, then $d \mid (a + b)$.
3. Show that for any integer n, the expression $n^2 + n$ is always divisible by 2.

Solutions

1. Use the division algorithm to find the unique integers q and r such that $1024 = q(37) + r$, where $0 \leq r < 37$.

   - Divide 1024 by 37 using long division or estimation.
   - Calculate $1024/37 \approx 27.67$, so the quotient q is 27.
   - Multiply $27 \times 37 = 999$.
   - Subtract this from the original number: $1024 - 999 = 25$.
   - Verify that $0 \leq 25 < 37$.

   Answer: q = 27, r = 25

2. Prove that if $d \mid a$ and $d \mid b$, then $d \mid (a + b)$.

   - By definition of divisibility, $d \mid a$ means $a = dk$ for some integer k.
   - Similarly, $d \mid b$ means $b = dm$ for some integer m.
   - The sum $a + b = dk + dm$.
   - Factor out d to get $a + b = d(k + m)$.
   - Since $k + m$ is an integer, d divides $a + b$ by definition.

   Answer: The sum is $d(k + m)$, which satisfies the definition of divisibility.

3. Show that for any integer n, the expression $n^2 + n$ is always divisible by 2.

   - Factor the expression as $n(n + 1)$.
   - Consider two cases: n is even or n is odd.
   - If n is even, $n = 2k$, so $n(n + 1) = 2k(2k + 1)$, which is a multiple of 2.
   - If n is odd, $n = 2k + 1$, so $n + 1 = 2k + 2 = 2(k + 1)$.
   - Then $n(n + 1) = (2k + 1) \cdot 2(k + 1)$, which is also a multiple of 2.
   - In both cases, the expression is divisible by 2.

   Answer: The product of two consecutive integers is always even.
