Multisets and the twelvefold way — a map of all counting problems

Practice problems

1. Calculate the number of ways to distribute 7 identical gold coins into 4 distinct treasure chests such that no chest is left empty.
2. How many ways can 4 distinct students be assigned to 3 identical study rooms such that no room is empty?
3. Find the number of ways to distribute 5 identical balls into 3 distinct boxes with no constraints.

Solutions

1. Calculate the number of ways to distribute 7 identical gold coins into 4 distinct treasure chests such that no chest is left empty.

   - Identify parameters: $n = 7$ (indistinguishable balls), $k = 4$ (distinguishable boxes).
   - Constraint: Surjective (no chest empty).
   - Use the formula $\binom{n-1}{k-1}$.
   - Compute $\binom{7-1}{4-1} = \binom{6}{3}$.
   - $\binom{6}{3} = \frac{6\times5\times4}{3\times2\times1} = 20$.

   Answer: 20

2. How many ways can 4 distinct students be assigned to 3 identical study rooms such that no room is empty?

   - Identify parameters: $n = 4$ (distinguishable balls), $k = 3$ (indistinguishable boxes).
   - Constraint: Surjective (no room empty).
   - This is the definition of the Stirling number of the second kind $S(4, 3)$.
   - The only partition of 4 into 3 non-empty sets is sizes $2, 1, 1$.
   - Number of ways to choose the pair is $\binom{4}{2} = 6$.

   Answer: 6

3. Find the number of ways to distribute 5 identical balls into 3 distinct boxes with no constraints.

   - Identify parameters: $n = 5$ (indistinguishable balls), $k = 3$ (distinguishable boxes).
   - Constraint: None.
   - Use the stars and bars formula $\binom{n+k-1}{k-1}$.
   - Compute $\binom{5+3-1}{3-1} = \binom{7}{2}$.
   - $\binom{7}{2} = \frac{7\times6}{2\times1} = 21$.

   Answer: 21
