Cardinality in the discrete world; Hilbert's hotel

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Practice problems

1. Let S be the set of all positive integers that are multiples of 3. Prove that $|S| = |\mathbb{N}|$.
2. If a hotel with $\aleph_0$ rooms is full, and a bus arrives with $\aleph_0$ guests, show that the manager can accommodate everyone by moving the current guest in room n to room $2n$. Which rooms become available for the new guests?
3. Prove that the set of all pairs of natural numbers, $\mathbb{N} \times \mathbb{N}$, is countable.

Solutions

1. Let S be the set of all positive integers that are multiples of 3. Prove that $|S| = |\mathbb{N}|$.

   - Define a function $f : \mathbb{N} \to S$.
   - Let $f(n) = 3n$.
   - Check for injection: if $3n_1 = 3n_2$, then $n_1 = n_2$.
   - Check for surjection: for any $m \in S$, m is a multiple of 3, so $m = 3k$ for some $k \in \mathbb{N}$. Thus $f(k) = m$.
   - Since f is a bijection, the sets have the same cardinality.

   Answer: $|S| = |\mathbb{N}|$

2. If a hotel with $\aleph_0$ rooms is full, and a bus arrives with $\aleph_0$ guests, show that the manager can accommodate everyone by moving the current guest in room n to room $2n$. Which rooms become available for the new guests?

   - The current guests move from $n \to 2n$.
   - The set of occupied rooms becomes the set of even numbers: $\{2, 4, 6, \dots\}$.
   - The set of empty rooms is the set of odd numbers: $\{1, 3, 5, \dots\}$.
   - The set of odd numbers is countably infinite, so it can accommodate the $\aleph_0$ new guests.

   Answer: The odd-numbered rooms become available.

3. Prove that the set of all pairs of natural numbers, $\mathbb{N} \times \mathbb{N}$, is countable.

   - Represent the pairs (i, j) as a 2D grid.
   - Order the pairs by the sum of their components $s = i + j$.
   - For a fixed sum s, there are only finitely many pairs (i, j).
   - List pairs with $s = 2$, then $s = 3$, then $s = 4$, and so on.
   - This creates a sequence that includes every pair exactly once, establishing a bijection with $\mathbb{N}$.

   Answer: $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$
