Functions revisited: injections, surjections, bijections and counting

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Practice problems

1. Let $|A| = 3$ and $|B| = 5$. How many total functions $f : A \to B$ exist, and how many of these are injections?
2. Let $|A| = 4$ and $|B| = 2$. How many surjective functions $f : A \to B$ exist?
3. If $|A| = n$ and $|B| = n$, prove that the number of injections is equal to the number of surjections.

Solutions

1. Let $|A| = 3$ and $|B| = 5$. How many total functions $f : A \to B$ exist, and how many of these are injections?

   - The total number of functions is m^n, where $m = 5$ and $n = 3$.
   - Calculation: $5^3 = 125$.
   - The number of injections is $P(m, n) = \frac{m!}{(m-n)!}$.
   - Calculation: $\frac{5!}{(5-3)!} = \frac{120}{2} = 60$.

   Answer: 125 total functions, 60 injections

2. Let $|A| = 4$ and $|B| = 2$. How many surjective functions $f : A \to B$ exist?

   - Use the surjection formula: $\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m - k)^n$ with $m = 2, n = 4$.
   - For $k = 0$: $\binom{2}{0}(2 - 0)^4 = 1 \cdot 16 = 16$.
   - For $k = 1$: $\binom{2}{1}(2 - 1)^4 = 2 \cdot 1 = 2$.
   - For $k = 2$: $\binom{2}{2}(2 - 2)^4 = 1 \cdot 0 = 0$.
   - Total: $16 - 2 + 0 = 14$.

   Answer: 14 surjections

3. If $|A| = n$ and $|B| = n$, prove that the number of injections is equal to the number of surjections.

   - The number of injections is $P(n, n) = n!$.
   - The number of surjections is $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)^n$.
   - From the theory of bijections, if $|A| = |B|$, any injection is automatically a surjection.
   - Therefore, the set of injections and the set of surjections are the same set (the set of bijections).
   - Both counts must equal $n!$.

   Answer: Both equal $n!$
