Order relations; posets and Hasse diagrams

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Practice problems

1. Let $S = \{a, b, c\}$. Is the relation $\subseteq$ on the power set $P(S)$ a partial order? If so, how many elements are in the middle level of its Hasse diagram?
2. Consider the set $S = \{1, 2, 3, 4, 5, 6\}$ under the divisibility relation. List all pairs of elements that are incomparable.
3. Does the relation R on the set of all integers $\mathbb{Z}$ defined by aRb if a divides b constitute a partial order? Explain why or why not.

Solutions

1. Let $S = \{a, b, c\}$. Is the relation $\subseteq$ on the power set $P(S)$ a partial order? If so, how many elements are in the middle level of its Hasse diagram?

   - Check reflexivity: Every set is a subset of itself, so $A \subseteq A$ holds.
   - Check antisymmetry: If $A \subseteq B$ and $B \subseteq A$, then $A = B$ by the definition of set equality.
   - Check transitivity: If $A \subseteq B$ and $B \subseteq C$, then every element of A is in B, and thus in C, so $A \subseteq C$.
   - The relation is a partial order. The power set has $2^3 = 8$ elements. The middle level consists of subsets with size 1 or 2. Specifically, the subsets of size 1 are the first level above the empty set, and subsets of size 2 are the next level. There are $\binom{3}{1} = 3$ and $\binom{3}{2} = 3$ such sets.

   Answer: Yes, it is a poset; there are 3 elements in the level of singletons and 3 in the level of pairs.

2. Consider the set $S = \{1, 2, 3, 4, 5, 6\}$ under the divisibility relation. List all pairs of elements that are incomparable.

   - Check all pairs (a, b) where $a \neq b$.
   - Pairs like $(2, 3)$ are incomparable because $2 \nmid 3$ and $3 \nmid 2$.
   - Pairs like $(2, 5)$, $(3, 4)$, $(3, 5)$, $(4, 5)$, $(4, 6)$, $(5, 6)$ are also incomparable.
   - Pairs like $(2, 4)$ are comparable because $2 \mid 4$.

   Answer: \(2, 3), (2, 5), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6)\

3. Does the relation R on the set of all integers $\mathbb{Z}$ defined by aRb if a divides b constitute a partial order? Explain why or why not.

   - Check reflexivity: $a \mid a$ is true for all $a \neq 0$. For $a = 0$, $0 \mid 0$ is usually defined as true in this context.
   - Check antisymmetry: For a partial order, aRb and bRa must imply $a = b$.
   - Consider $a = 1$ and $b = -1$. 1 divides -1 (since $-1 = 1 \cdot -1$) and -1 divides 1 (since $1 = -1 \cdot -1$).
   - However, $1 \neq -1$. Therefore, the relation is not antisymmetric.

   Answer: No, it is not a partial order because it fails antisymmetry.
