Equivalence relations and partitions — the same, seen two ways

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Practice problems

1. Let $S = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$. Is R an equivalence relation? Explain why or why not.
2. Consider the set $S = \{1, 2, 3, 4, 5, 6\}$ and the partition $P = \{\{1, 3, 5\}, \{2, 4\}, \{6\}\}$. List the ordered pairs of the equivalence relation R induced by this partition.
3. Prove that for any integer $n > 1$, the relation $a \equiv b \pmod{n}$ is an equivalence relation on the set of integers $\mathbb{Z}$.

Solutions

1. Let $S = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$. Is R an equivalence relation? Explain why or why not.

    - Check reflexivity: $(1, 1), (2, 2), (3, 3)$ are all in R. Reflexive.
    - Check symmetry: $(1, 2) \in R$ and $(2, 1) \in R$; $(2, 3) \in R$ and $(3, 2) \in R$. Symmetric.
    - Check transitivity: $(1, 2) \in R$ and $(2, 3) \in R$, but $(1, 3) \notin R$.
    - Since transitivity fails, it is not an equivalence relation.

    Answer: No, it is not an equivalence relation because it is not transitive.

2. Consider the set $S = \{1, 2, 3, 4, 5, 6\}$ and the partition $P = \{\{1, 3, 5\}, \{2, 4\}, \{6\}\}$. List the ordered pairs of the equivalence relation R induced by this partition.

    - Elements in the same subset are related. For $\{1, 3, 5\}$, we have $(1, 1), (3, 3), (5, 5), (1, 3), (3, 1), (1, 5), (5, 1$
    - For $\{2, 4\}$, we have $(2, 2), (4, 4), (2, 4), (4, 2)$.
    - For $\{6\}$, we have $(6, 6)$.
    - The relation R is the union of all these pairs.

    Answer: $R = \{(1, 1), (3, 3), (5, 5), (1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3), (2, 2), (4, 4), (2, 4), (4, 2), (6, 6)\}$

3. Prove that for any integer $n > 1$, the relation $a \equiv b \pmod{n}$ is an equivalence relation on the set of integers $\mathbb{Z}$.

    - Reflexivity: $a - a = 0$. Since $n \cdot 0 = 0$, n divides $a - a$, so $a \equiv a \pmod{n}$.
    - Symmetry: If $a \equiv b \pmod{n}$, then $a - b = kn$ for some integer k. Then $b - a = (-k)n$. Since $-k$ is an integer, n divides $b - a$, so $b \equiv a \pmod{n}$.
    - Transitivity: If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a - b = kn$ and $b - c = mn$. Adding these gives $(a - b) + (b - c) = kn + mn$, so $a - c = (k + m)n$. Since $k + m$ is an integer, n divides $a - c$, so $a \equiv c \pmod{n}$.

    Answer: The relation is reflexive, symmetric, and transitive, therefore it is an equivalence relation.
