Module 0 review + self-test

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Practice problems

1. Prove that in any group of 6 people, there are either 3 people who all know each other or 3 people who are all strangers to each other.
2. Simplify the boolean expression $(P \vee Q) \wedge (P \vee \neg Q)$ using logical equivalences.
3. Prove by induction that for all $n \geq 1$, the sum $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Solutions

1. Prove that in any group of 6 people, there are either 3 people who all know each other or 3 people who are all strangers to each other.

   - Pick one person, A. There are 5 other people.
   - By the pigeonhole principle, A must either know at least 3 people or not know at least 3 people.
   - Case 1: A knows B, C, and D. If any two of B, C, D know each other, we have a group of 3 friends. If none of them know each other, B, C, and D are 3 strangers.
   - Case 2: A does not know B, C, and D. If any two of B, C, D are strangers, we have a group of 3 strangers. If all of them know each other, B, C, and D are 3 friends.

   Answer: By the pigeonhole principle and case analysis, a group of 6 always contains a 3-clique or a 3-independent set.

2. Simplify the boolean expression $(P \lor Q) \land (P \lor \neg Q)$ using logical equivalences.

   - Use the distributive law in reverse: $(P \lor Q) \land (P \lor \neg Q) \equiv P \lor (Q \land \neg Q)$.
   - Recognize that $Q \land \neg Q$ is a contradiction, which is always False.
   - The expression becomes $P \lor \text{False}$.
   - Any statement OR False is simply the statement itself.

   Answer: P

3. Prove by induction that for all $n \geq 1$, the sum $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

   - Base case: For $n = 1$, $1 = \frac{1(1+1)}{2} = 1$. True.
   - Inductive step: Assume $S_k = \frac{k(k+1)}{2}$.
   - For $n = k + 1$, $S_{k+1} = S_k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$.
   - Factor out $(k + 1)$: $(k + 1)(\frac{k}{2} + 1) = (k + 1)(\frac{k+2}{2}) = \frac{(k+1)(k+2)}{2}$.
   - This matches the formula for $n = k + 1$.

   Answer: \fracn(n+1)2
