The pigeonhole principle II — harder gems

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Practice problems

1. Prove that in any set of 10 integers, there are two whose sum or difference is divisible by 17.
2. Prove that in any set of $n+1$ integers, there exist two integers x and y such that n divides $x-y$.
3. Show that any sequence of n integers contains a contiguous subsequence whose sum is divisible by n.

Solutions

1. Prove that in any set of 10 integers, there are two whose sum or difference is divisible by 17.

   - Consider the remainders of the 10 integers modulo 17. The possible remainders are $0, 1, \ldots, 16$.
   - Group these remainders into 9 holes: $H_0 = \{0\}$, $H_1 = \{1, 16\}$, $H_2 = \{2, 15\}$, $H_3 = \{3, 14\}$, $H_4 = \{4, 13\}$, $H_5 = \{5, 12\}$, $H_6 = \{6, 11\}$, $H_7 = \{7, 10\}$, $H_8 = \{8, 9\}$.
   - Since there are 10 integers and only 9 holes, at least two integers must fall into the same hole by the pigeonhole principle.
   - If two integers are in H_0, their difference is $0 - 0 = 0 \pmod{17}$.
   - If two integers are in H_k for $k > 0$, they are either the same remainder (difference is $0 \pmod{17}$) or they are the two different values in the set (sum is $k + (17 - k) = 17 \equiv 0 \pmod{17}$).

   Answer: By grouping remainders into pairs that sum to 17, 10 integers must occupy at least one of the 9 holes, forcing a sum or difference divisible by 17.

2. Prove that in any set of $n+1$ integers, there exist two integers x and y such that n divides $x - y$.

   - Let the $n + 1$ integers be the pigeons.
   - Let the possible remainders modulo n be the holes. There are exactly n such remainders: $0, 1, \ldots, n - 1$.
   - By the pigeonhole principle, since there are $n + 1$ integers and n remainders, at least two integers x and y must have the same remainder r modulo n.
   - Thus, $x = q_1 n + r$ and $y = q_2 n + r$.
   - The difference $x - y = (q_1 - q_2)n$, which is clearly divisible by n.

   Answer: Two integers with the same remainder modulo n have a difference divisible by n.

3. Show that any sequence of n integers contains a contiguous subsequence whose sum is divisible by n.

   - Define the prefix sums $S_k = a_1 + a_2 + \cdots + a_k$ for $k = 1, \ldots, n$.
   - Consider these n sums modulo n.
   - If any $S_k \equiv 0 \pmod{n}$, then the subsequence from a_1 to a_k is divisible by n.
   - If no $S_k \equiv 0 \pmod{n}$, then the n sums must take values from the set $\{1, 2, \ldots, n - 1\}$.
   - Since there are n sums and only $n-1$ possible values, two sums must be congruent: $S_i \equiv S_j \pmod{n}$ for some $i < j$.
   - Then $S_j - S_i = a_{i+1} + \cdots + a_j \equiv 0 \pmod{n}$.

   Answer: Either a prefix sum is $0 \pmod{n}$ or two prefix sums are equal $\pmod{n}$, ensuring a contiguous sum divisible by n.