Proof techniques tour

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Practice problems

1. Use a direct proof to show that if n and m are odd integers, then their sum $n + m$ is an even integer.
2. Prove by contrapositive: If n^2 is odd, then n is odd.
3. Use induction to prove that for all $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solutions

1. Use a direct proof to show that if n and m are odd integers, then their sum $n+m$ is an even integer.

   - Assume n and m are odd, so $n = 2k+1$ and $m = 2j+1$ for some integers k, j.
   - Sum them: $n+m = (2k+1)+(2j+1) = 2k+2j+2$.
   - Factor out a 2: $n+m = 2(k+j+1)$.
   - Since $k+j+1$ is an integer, $n+m$ is even by definition.

   Answer: The sum is $2(k+j+1)$, which is even.

2. Prove by contrapositive: If n^2 is odd, then n is odd.

   - State the contrapositive: If n is not odd (even), then n^2 is not odd (even).
   - Assume n is even, so $n = 2k$.
   - Square n: $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
   - Since $2k^2$ is an integer, n^2 is even.
   - The contrapositive is true, so the original statement is true.

   Answer: The contrapositive 'if n is even, then n^2 is even' is true, proving the claim.

3. Use induction to prove that for all $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   - Base case $n = 1$: $1^2 = 1$ and $\frac{1(2)(3)}{6} = 1$. True.
   - Inductive step: Assume $S_k = \frac{k(k+1)(2k+1)}{6}$.
   - Add $(k+1)^2$: $S_{k+1} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$.
   - Factor out $(k+1)$: $\frac{k+1}{6}[k(2k+1)+6(k+1)] = \frac{k+1}{6}[2k^2+7k+6]$.
   - Factor the quadratic: $2k^2+7k+6 = (k+2)(2k+3)$.
   - Result: $S_{k+1} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$. True.

   Answer: The formula holds for $n = 1$ and the inductive step is verified.
