

Sets of Measure Zero and Lebesgue's Criterion

Lesson 88 · Module 7 — The Riemann Integral · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Let $f(x)$ be a bounded function on $[0, 1]$ that is discontinuous only at the points $x_n = (1/2)^n$ for $n = 1, 2, 3, \dots$. Is f Riemann integrable?
2. Consider the function $f(x) = 1$ if x is a rational in $[0, 1]$ and $f(x) = 0$ if x is irrational. Use Lebesgue's criterion to explain why it is not Riemann integrable.
3. If f and g are Riemann integrable on $[a, b]$, prove that $f + g$ is also Riemann integrable using Lebesgue's criterion.

Solutions

1. Let $f(x)$ be a bounded function on $[0, 1]$ that is discontinuous only at the points $x_n = (1/2)^n$ for $n = 1, 2, 3, \dots$. Is f Riemann integrable?

- Identify the set of discontinuities $S = \{(1/2)^n : n \in \mathbb{N}\}$.
- Observe that S is a countable set.
- Recall that every countable set has measure zero.
- Since the set of discontinuities has measure zero, f is continuous almost everywhere.
- By Lebesgue's criterion, f is Riemann integrable.

Answer: Yes, it is Riemann integrable.

2. Consider the function $f(x) = 1$ if x is a rational in $[0, 1]$ and $f(x) = 0$ if x is irrational. Use Lebesgue's criterion to explain why it is not Riemann integrable.

- The function is discontinuous at every point $x \in [0, 1]$.
- The set of discontinuities is $S = [0, 1]$.
- The measure of the interval $[0, 1]$ is $1 - 0 = 1$.
- Since $1 \neq 0$, the set of discontinuities does not have measure zero.
- Therefore, f is not continuous almost everywhere and is not Riemann integrable.

Answer: Not integrable because the set of discontinuities has measure 1.

3. If f and g are Riemann integrable on $[a, b]$, prove that $f + g$ is also Riemann integrable using Lebesgue's criterion.

- Let D_f be the set of discontinuities of f and D_g be the set of discontinuities of g .
- Since f and g are integrable, D_f and D_g both have measure zero.
- The sum $f + g$ is discontinuous only at points where either f or g (or both) are discontinuous.
- Thus, the set of discontinuities of $f + g$ is $D_{f+g} \subset D_f \cup D_g$.
- The union of two sets of measure zero also has measure zero.
- Since D_{f+g} is a subset of a set of measure zero, it also has measure zero.
- By Lebesgue's criterion, $f + g$ is Riemann integrable.

Answer: The set of discontinuities of $f + g$ is a subset of the union of the discontinuities of f and g , which has measure zero.