

# Improper Integrals Rigorously; Comparison Tests Proved

Lesson 87 · Module 7 — The Riemann Integral · Real Analysis · [dailymaths.uk/analysis](http://dailymaths.uk/analysis)

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## Practice problems

1. Determine if the integral  $\int_1^\infty \frac{1}{x^2+x} dx$  converges or diverges. If it converges, find its value.
2. Use the Comparison Test to determine the convergence of  $\int_1^\infty \frac{\sin^2(x)}{x^3} dx$ .
3. Evaluate the improper integral  $\int_0^1 \frac{1}{x^2} dx$  and explain the result.

## Solutions

1. Determine if the integral  $\int_1^\infty \frac{1}{x^2+x} dx$  converges or diverges. If it converges, find its value.

- Compare the integrand to  $\frac{1}{x^2}$ . Since  $x^2 + x > x^2$  for  $x \geq 1$ , we have  $0 < \frac{1}{x^2+x} < \frac{1}{x^2}$ .
- We know  $\int_1^\infty \frac{1}{x^2} dx$  converges (p-test with  $p = 2$ ), so the original integral converges by the Comparison Test.
- To find the value, use partial fractions:  $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ .
- Integrate:  $\int_1^t (\frac{1}{x} - \frac{1}{x+1}) dx = [\ln x - \ln(x+1)]_1^t = \ln(\frac{t}{t+1}) - \ln(\frac{1}{2})$ .
- Take the limit as  $t \rightarrow \infty$ :  $\lim_{t \rightarrow \infty} \ln(\frac{t}{t+1}) = \ln(1) = 0$ .
- The result is  $0 - \ln(1/2) = \ln(2)$ .

**Answer:**  $\ln(2)$

2. Use the Comparison Test to determine the convergence of  $\int_1^\infty \frac{\sin^2(x)}{x^3} dx$ .

- Identify the bounds of the numerator:  $0 \leq \sin^2(x) \leq 1$  for all  $x$ .
- Divide by  $x^3$  to get the inequality:  $0 \leq \frac{\sin^2(x)}{x^3} \leq \frac{1}{x^3}$ .
- Check the convergence of the larger function:  $\int_1^\infty \frac{1}{x^3} dx$  is a p-integral with  $p = 3$ .
- Since  $3 > 1$ , the integral  $\int_1^\infty \frac{1}{x^3} dx$  converges.
- By the Comparison Test, the original integral also converges.

**Answer:** Converges

3. Evaluate the improper integral  $\int_0^1 \frac{1}{x^2} dx$  and explain the result.

- The function is unbounded at  $x = 0$ , so we set up the limit:  $\lim_{\epsilon \rightarrow 0^+} \int_\epsilon^1 x^{-2} dx$ .
- Find the antiderivative:  $\int x^{-2} dx = -x^{-1} = -\frac{1}{x}$ .
- Evaluate the definite integral:  $[-\frac{1}{x}]_\epsilon^1 = -1 - (-\frac{1}{\epsilon}) = \frac{1}{\epsilon} - 1$ .
- Take the limit as  $\epsilon \rightarrow 0^+$ :  $\lim_{\epsilon \rightarrow 0^+} (\frac{1}{\epsilon} - 1) = \infty$ .
- Since the limit is infinite, the integral diverges.

**Answer:** Diverges