

The integral MVT; averages justified

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Practice problems

1. Find the average value of $f(x) = \sqrt{x}$ on the interval $[0, 4]$ and find all values of $c \in [0, 4]$ such that $f(c)$ equals this average value.
2. For $f(x) = e^x$ on $[0, 1]$, find the value of c that satisfies the Mean Value Theorem for Integrals.
3. Suppose f is continuous on $[a, b]$. Prove that if $f(x) > 0$ for all $x \in [a, b]$, then the average value f_{avg} must also be strictly positive.

Solutions

1. Find the average value of $f(x) = \sqrt{x}$ on the interval $[0, 4]$ and find all values of $c \in [0, 4]$ such that $f(c)$ equals this average value.

- Compute the integral: $\int_0^4 x^{1/2} dx = [\frac{2}{3}x^{3/2}]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$.
- Calculate average value: $f_{avg} = \frac{1}{4-0} \cdot \frac{16}{3} = \frac{4}{3}$.
- Set $f(c) = \frac{4}{3}$: $\sqrt{c} = \frac{4}{3}$.
- Solve for c : $c = (\frac{4}{3})^2 = \frac{16}{9}$.

Answer: $f_{avg} = \frac{4}{3}$, $c = \frac{16}{9}$

2. For $f(x) = e^x$ on $[0, 1]$, find the value of c that satisfies the Mean Value Theorem for Integrals.

- Compute the integral: $\int_0^1 e^x dx = [e^x]_0^1 = e - 1$.
- Calculate average value: $f_{avg} = \frac{e-1}{1-0} = e - 1$.
- Set $f(c) = e - 1$: $e^c = e - 1$.
- Solve for c : $c = \ln(e - 1)$.

Answer: $c = \ln(e-1)$

3. Suppose f is continuous on $[a, b]$. Prove that if $f(x) > 0$ for all $x \in [a, b]$, then the average value f_{avg} must also be strictly positive.

- Since $f(x) > 0$ and is continuous on a closed interval, by the Extreme Value Theorem, it has a minimum $m > 0$.
- The integral $\int_a^b f(x) dx \geq m(b - a)$.
- Since $m > 0$ and $b - a > 0$, the integral is strictly positive.
- The average value $f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$ is therefore a positive number divided by a positive number, which is strictly positive.

Answer: $f_{avg} > 0$