

# Properties of the integral — proved

Lesson 82 · Module 7 — The Riemann Integral · Real Analysis · [dailymaths.uk/analysis](http://dailymaths.uk/analysis)

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## Practice problems

1. Given that  $\int_0^1 f(x) dx = 5$  and  $\int_0^1 g(x) dx = -2$ , compute  $\int_0^1 (3f(x) - 4g(x)) dx$ .
2. Let  $f(x) = x$  for  $x \in [0, 1]$  and  $f(x) = 2$  for  $x \in (1, 2]$ . Use interval additivity to find  $\int_0^2 f(x) dx$ .
3. Prove that the function  $h(x) = x^2$  if  $x \in \mathbb{Q}$  and  $h(x) = -x^2$  if  $x \notin \mathbb{Q}$  is Riemann integrable on  $[0, 1]$ .

## Solutions

1. Given that  $\int_0^1 f(x) dx = 5$  and  $\int_0^1 g(x) dx = -2$ , compute  $\int_0^1 (3f(x) - 4g(x)) dx$ .

- Use linearity to split the integral:  $3 \int_0^1 f(x) dx - 4 \int_0^1 g(x) dx$ .
- Substitute the given values:  $3(5) - 4(-2)$ .
- Calculate the result:  $15 + 8 = 23$ .

**Answer:** 23

2. Let  $f(x) = x$  for  $x \in [0, 1]$  and  $f(x) = 2$  for  $x \in (1, 2]$ . Use interval additivity to find  $\int_0^2 f(x) dx$ .

- Split the integral at  $x = 1$ :  $\int_0^1 x dx + \int_1^2 2 dx$ .
- Compute the first part:  $[x^2/2]_0^1 = 1/2$ .
- Compute the second part:  $[2x]_1^2 = 4 - 2 = 2$ .
- Sum the results:  $1/2 + 2 = 2.5$ .

**Answer:** 2.5

3. Prove that the function  $h(x) = x^2$  if  $x \in \mathbb{Q}$  and  $h(x) = -x^2$  if  $x \notin \mathbb{Q}$  is Riemann integrable on  $[0, 1]$ .

- Identify two integrable functions that bound  $h(x)$ .
- Note that  $-x^2 \leq h(x) \leq x^2$  for all  $x \in [0, 1]$ .
- Since  $g(x) = -x^2$  and  $k(x) = x^2$  are continuous, they are integrable.
- By the comparison property, since  $h(x)$  is trapped between two integrable functions,  $h(x)$  is also integrable.

**Answer:** Integrable by comparison property